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A.S. SILBERGLEIT and G.M. KEISER

On eddy currents from moving point sources of magnetic field in the Gravity Probe B experiment

1. Introduction

In the Gravity Probe B experiment [1], a superconducting rotating ball will be orbiting the Earth for over a year, and the drift of its axis resulting from two General Relativity effects will be measured. The drift is expected to be very small (0.042 arc-sec/year and 6.6 arc-sec/year in the East-West and North-South directions, respectively) and the measurement accuracy should be very high, so the classical (non-relativistic) torques causing the drift must be either eliminated or carefully accounted for. In particular, there will be quantum-size sources of magnetic field (fluxons) on the surface of the superconducting rotor which induce eddy currents and thus energy dissipation in surrounding normal metals. Consequently, differential damping torques are produced which must be estimated.

In this paper we give such estimates by solving explicitly two corresponding model boundary value problems in plane geometry (they may also be of interest for other applications). We are able to avoid complications of the spherical case, imminent for the GP–B experiment, since the gap between the rotor surface and normal metals around it is extremely small compared to the rotor's radius.

2. The problem for a conducting half-space

Suppose that a metal with electrical conductivity σ and magnetic constant $\mu = \mu_0$ occupies the half-space $z' > d$ of a Cartesian coordinate system $\{x', y', z'\}$. The plane $z = 0$ is a superconductor's surface in which a fluxon and antifluxon move with a constant velocity $v\mathcal{L}c$ along the x'-axis; the layer $0 < z' < d$ is a dielectric gap. According to [2, **10.00**], the set of governing equations and boundary conditions for the *quasi-stationary* magnetic induction $\mathbf{B}$ in the dimensionless coordinates $x = (x' - x'_f(t))/d$, $y = (y' - y'_f)/d$, $z = z'/d$ co-moving with the sources is

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = 0, \quad 0 < z < 1, \quad |x|, |y| < \infty, \tag{1}$$

$$B_z|_{z=0} = \frac{\Phi_0}{d^2}\left[\delta(x)\delta(y) - \delta(x - x_0)\delta(y - y_0)\right], \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \Delta \mathbf{B} = -\kappa \frac{\partial \mathbf{B}}{\partial x}, \quad z > 1, \quad |x|, |y| < \infty, \quad \mathbf{B}|_{z=1+0} = \mathbf{B}|_{z=1-0}. \tag{3}$$

Here Φ_0 is the flux of a point source (for GP–B, $\Phi_0 = h/2e$ is the magnetic flux quantum), $x'_{f,a}(t) = x'_{f,a} + vt$ and $y'_{f,a}$ are the coordinates of the fluxon/antifluxon, $x_0 = (x_a - x_f)/d$, $y_0 = (y_a - y_f)/d$, the rest of the notations are standard, and we have omitted the obvious decay conditions at infinity. The only dimensionless parameter of the problem is $\kappa = \sigma\mu_0 vd$.

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Problem (1)–(3) may be solved by means of the Fourier integral transform in the x and y variables, which leads to rather cumbersome double Fourier integrals preventing further analytic calculations and hiding the physical meaning of the result. Instead, we look for a perturbative solution as a power series in the small parameter κ, $\kappa \lesssim 0.02$, for the GP-B conditions, that is,

$$\mathbf{B}(x,y,z) = \mathbf{B}^{(0)}(x,y,z) + \kappa \mathbf{B}^{(1)}(x,y,z) + \kappa^2 \mathbf{B}^{(2)}(x,y,z) + \cdots, \tag{4}$$

where the $\mathbf{B}^{(k)}$, $k = 0, 1, 2, \ldots$, are to be determined successively from the sequence of problems implied by (1)–(4), namely

$$\begin{gathered} \nabla \cdot \mathbf{B}^{(k)} = 0, \quad \nabla \times \mathbf{B}^{(k)} = 0, \quad z < 1, \\ B_z^{(k)}|_{z=0} = \frac{\Phi_0}{d^2}\left[\delta(x)\delta(y) - \delta(x-x_0)\delta(y-y_0)\right]\delta_{k0}, \end{gathered} \tag{5}$$

$$\nabla \cdot \mathbf{B}^{(k)} = 0, \ \Delta \mathbf{B}^{(k)} = -(1-\delta_{k0})\frac{\partial \mathbf{B}^{(k-1)}}{\partial x}, \ z > 1, \ \mathbf{B}^{(k)}|_{z=1+0} = \mathbf{B}^{(k)}|_{z=1-0}. \tag{6}$$

The corresponding eddy current density in the conductor is ([2, **10.00**])

$$\mathbf{j} = \frac{1}{\mu_0 d}\nabla \times \mathbf{B} = (\kappa \mathbf{j}^{(1)} + \kappa^2 \mathbf{j}^{(2)} + \cdots), \quad \mathbf{j}^{(k)} = \frac{1}{\mu_0 d}\nabla \times \mathbf{B}^{(k)}, \quad z > 1, \tag{7}$$

where we have taken into account the fact that $\nabla \times \mathbf{B}^{(0)} = 0$ in the whole half-space $z > 0$. Naturally, the current density is of the first order in κ, since without a conductor ($\kappa = 0$) there is no current at all. Our aim is to determine $\mathbf{j}^{(1)}$ and then to calculate the dissipated power to the first non-vanishing order.

The 'unperturbed' field $\mathbf{B}^{(0)}$ is that of the two point sources in a half-space without a conductor. It satisfies the Laplace equation in the whole half-space $z > 0$ with the boundary condition (2) and therefore is given by

$$\mathbf{B}^{(0)} = \nabla \psi^{(0)}, \quad \psi^{(0)}(x,y,z) = -\frac{\Phi_0}{2\pi d^2}\left(\frac{1}{R} - \frac{1}{R_0}\right), \quad z > 0, \tag{8}$$

where $R = \sqrt{x^2+y^2+z^2}$ and $R_0 = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$.

To find $\mathbf{j}^{(1)}$ using (7), we do not need $\mathbf{B}^{(1)}$ itself (from (6) with $k = 1$), but rather its curl in the conductor. Surprisingly, the latter may be determined without completely solving the boundary value problem: from (5) and (6) we can derive a representation of the form

$$\mathbf{B}^{(1)} = \nabla\left(\psi^{(1)} - \frac{x}{2}\psi^{(0)}\right) + \nabla \times \mathbf{A}^{(1)} + \psi^{(0)}\mathbf{e}_x, \quad z > 1, \tag{9}$$

where $\psi^{(1)}$ and $\mathbf{A}^{(1)}$ are harmonic functions in $z > 1$, and $\mathbf{A}^{(1)}$ is divergenceless there. Therefore, the first two terms in (9) are curl-less, so that we have (see (7))

$$\nabla \times \mathbf{B}^{(1)} = \nabla\psi^{(0)} \times \mathbf{e}_x = \mathbf{B}^{(0)} \times \mathbf{e}_x, \quad \mathbf{j} = \kappa \mathbf{j}^{(1)} + O(\kappa^2) = \sigma v\, \mathbf{B}^{(0)} \times \mathbf{e}_x + O(\kappa^2). \tag{10}$$

The last result has a clear physical meaning: when a conductor moves with velocity $\mathbf{v}^{(0)}$ in an external magnetic field $\mathbf{B}^{(0)}$, the induced current density in it is given by the Lorentz formula $\mathbf{j} = \sigma \mathbf{v}^{(0)} \times \mathbf{B}^{(0)}$. In our case $\mathbf{v}^{(0)} = -v\,\mathbf{e}_x$, and, as the conductivity is small, the induced magnetic field may be neglected compared to the 'external' field $\mathbf{B}^{(0)}$, which amounts exactly to (10).

We now evaluate the power dissipated by a fluxon–antifluxon pair as

$$P_{fa} = \frac{d^3}{\sigma} \int\limits_{z>1} j^2\, dV \le 4P_f = \frac{d^3}{\sigma} \int\limits_{z>1} j_f^2\, dV \simeq \sigma v^2 d^3 \int\limits_{z>1} \left[\left(B_{f,y}^{(0)}\right)^2 + \left(B_{f,z}^{(0)}\right)^2 \right] dV,$$

where P_f is the power dissipated by a single fluxon, $\mathbf{B}_f^{(0)} \sim \nabla(1/R)$ (see (8)), and $\simeq$ means that we calculate to the first non-vanishing order in κ. By (8), this leads to the simplified and rather conservative estimate

$$P_{fa} \le 4P_f = \frac{3}{4\pi} \frac{\sigma v^2 \Phi_0^2}{d}. \tag{11}$$

Typically, the value of P_{fa} is at least twice smaller than this upper bound.

3. The problem for a strongly conducting thin layer

We now turn to the second problem, in which the conducting half-space is replaced by a conducting layer of thickness d_0 in the domain $d < z' < d + d_0$. The layer is relatively thin, $d_0 \mathcal{L} d$, and its conductivity is high, so that κ is no longer small (for the GP–B, $d_0 \sim 0.1d$ and $\kappa \sim 30$).

Consequently, outside the layer the magnetic field satisfies (1) with the boundary condition (2), while in the layer equations (3) are valid; the appropriate matching conditions are imposed at both surfaces of the layer (from here on we again use the unprimed dimensionless co-moving coordinates). The solution of this problem is very complicated, so another physically sound simplification of the model is required. The proper one is to replace the thin highly conducting layer of a *finite* thickness by an *infinitely thin* sheet carrying current of some surface density $\mathbf{j}^s = \mathbf{j}^s(x, y)$, and to formulate the matching conditions for the magnetic field at $z = 1$. The latter are known to be ([2], **7.21**)

$$B_z|_{z=1+0} = B_z|_{z=1-0}, \quad \mathbf{e}_z \times \left(\mathbf{B}|_{z=1+0} - \mathbf{B}|_{z=1,-0}\right) = \mu_0 \mathbf{j}^s; \tag{12}$$

thus, to solve the problem, we only need to relate $\mathbf{j}^s$ to $\mathbf{B}$. For this we introduce the vector potential $\mathbf{A}$ defined by $\mathbf{B} = \nabla \times \mathbf{A}$, $\nabla \cdot \mathbf{A} = 0$, and note that the induction equation, which is valid in a real three-dimensional layer, implies that $\mathbf{j} = \sigma v\, \partial \mathbf{A}/\partial x$. Hence, for a *surface* current density we can write

$$\mathbf{j}^s(x, y) = \sigma v d_0 \left. \frac{\partial \mathbf{A}}{\partial x} \right|_{z=1}. \tag{13}$$

The verification of the matching condition (12) and (13) is provided by the asymptotic integration of the complete three-dimensional problem from the asymptotic theory of 'thin' bodies (plates, shells; cf. [3]).

In terms of $\mathbf{A}$ we still have a very cumbersome vector boundary value problem, which we simplify by setting $A_z \equiv 0$ so that $\nabla \cdot \mathbf{A} = \partial A_x/\partial x + \partial A_y/\partial y = 0,\ z > 0$. The latter is satisfied if $A_x = -\partial \Pi/\partial y$ and $A_y = \partial \Pi/\partial x$, where $\Pi = \Pi(x, y, z)$ is a new function to be determined. By this and (13),

$$j_x^s = -\sigma v d_0 \left.\frac{\partial^2 \Pi}{\partial x \partial y}\right|_{z=1}, \quad j_y^s = \sigma v d_0 \left.\frac{\partial^2 \Pi}{\partial x^2}\right|_{z=1}. \tag{14}$$

Now it is not difficult to check that all the relevant equations are valid provided that $\Pi(x, y, z)$ solves the problem

$$\Delta \Pi = 0,\ z > 0,\ z \neq 1,\ -\left.\frac{\partial^2 \Pi}{\partial z^2}\right|_{z=0} = \frac{\Phi_0}{d^2}\left[\delta(x)\delta(y) - \delta(x - x_0)\delta(y - y_0)\right], \tag{15}$$

$$\left.\Big\langle \Pi \Big\rangle\right|_{z=1} = 0, \quad \left.\Big\langle \frac{\partial \Pi}{\partial z} \Big\rangle\right|_{z=1} = -\kappa_0 \left.\frac{\partial \Pi}{\partial x}\right|_{z=1},$$

where the angle brackets stand for the jump of the quantity inside them at $z = 1$, and $\kappa_0 = \kappa\, d_0/d = \sigma \mu_0 v d_0$. With the exception of the second derivative in the boundary condition at $z = 0$, this scalar problem is a standard one; its solution for $z \geq 1$ by means of the Fourier integral transform in x and y is ($\hat{f}(\lambda, \nu, \ldots)$, which denotes the corresponding Fourier image of $f(x, y, \ldots)$):

$$\hat{\Pi}(\lambda, \nu, z) = D(\lambda, \nu)\, e^{-\gamma z}, \quad \gamma(\lambda, \nu) = \sqrt{\lambda^2 + \nu^2} \geq 0, \tag{16}$$

$$D(\lambda, \nu) = -\frac{\Phi_0}{d^2 \gamma}\, \frac{1 - e^{-i(\lambda x_0 + \nu y_0)}}{\gamma - i(\kappa_0/2)\lambda(1 - e^{-\gamma})}, \quad z \geq 1.$$

From (16) and (14) we obtain

$$\frac{\hat{j}_x^s(\lambda, \nu)}{\sigma v d_0} = \lambda \nu D(\lambda, \nu) e^{-\gamma}, \quad \frac{\hat{j}_y^s(\lambda, \nu)}{\sigma v d_0} = -\lambda^2 D(\lambda, \nu) e^{-\gamma}. \tag{17}$$

To compute the dissipation rate P_{fa}, we use the Parseval identity and (17), which yield

$$P_{fa} = \frac{d^2}{\sigma d_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\left(\hat{j}_x^s(\lambda, \nu)\right)^2 + \left(\hat{j}_y^s(\lambda, \nu)\right)^2\right] d\lambda d\nu$$

$$= \sigma v^2 d_0 d^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \lambda^2 \gamma^2 e^{-2\gamma} |D(\lambda, \nu)|^2\, d\lambda d\nu.$$

Substituting (16) into this and using the inequality $\left|1 - \exp(-iQ)\right|^2 = 4\sin^2(0.5Q) \leq 4$ for simplification, we arrive at the desired estimate (cf. (11))

$$P_{fa} \leq 4P_f = \frac{4\sigma v^2 d_0 \Phi_0^2}{d^2} C(\kappa_0), \tag{18}$$

where

$$C(\kappa_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\lambda^2 e^{-2\gamma}}{\gamma^2 + (\kappa_0/2)^2\lambda^2(1-e^{-2\gamma})^2}\, d\lambda d\nu, \quad \gamma(\lambda,\nu) = \sqrt{\lambda^2+\nu^2} \geq 0.$$

The double integral representing the coefficient $C(\kappa_0)$ may be calculated explicitly as a combination of elementary functions, and also allows for a universal estimate $C(\kappa_0) < C(0) = \pi/4$, which, being introduced into (18), provides a final bound on P_{fa}, given by

$$P_{fa} \leq 4P_f < \frac{\pi\sigma v^2 d_0 \Phi_0^2}{d^2}. \tag{19}$$

For the GP–B conditions, numerical estimates of the corresponding torques and drift rates based on (11) and (19) show the latter to be smaller than those from some other differential damping torques [4], and thus do not endanger the experiment accuracy.

References

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Gravity Probe B, HEPL, Department of Physics, Stanford University, Stanford, CA 94305–4085, USA