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The Stanford Relativity Gyroscope Experiment (F): Correction to the Predicted Geodetic Precession of the Gyroscope Resulting from the Earth's Oblateness

J. V. Breakwell

1. INTRODUCTION

In 1968 D. C. Wilkins [1] and R. F. O'Connell [2] independently found that the oblateness of the earth modifies the geodetic precession of a gyroscope by an amount of order J_2 times Ω_G, where Ω_G is the 6.9 arc-sec/yr geodetic precession calculated by L. I. Schiff. This correction, being of order several milliarc-sec/yr, has to be taken into account in analyzing the data from the experiment.

The original investigations gave the magnitude $(\Omega_G)_{J_2}$ of the precession rate due to J_2 at a point $\mathbf{r}$ in space. For comparison with experiment, what is needed is not the precession rate at a point but its time integral over an orbit. Now the oblateness modifies not only the relativity effect but also the shape of the orbit (which deviates from an exact Keplerian ellipse by distances up to several miles). It also makes the orbit-plane regress.

Hence instead of simply integrating the original expression of Wilkins and O'Connell around a circular or elliptic orbit it is necessary to integrate a complicated vector function of the velocity and gravitational field over time.

The problem was solved for nearly circular orbits by Wilkins [3] in 1970 and by Barker and O'Connell [4] in 1972 using very different methods. More recently, F. R. Hoots and P. M. Fitzpatrick [5] have extended the investigation to highly elliptic orbits. All of these investigations depend on rather elaborate analytical procedures. Here I offer a simple treatment of the effect in a near-circular orbit based on a treatment of such orbits due originally to Laplace, and revived in the 1950's by R. E. Roberson [6] and by C. M. Petty and myself [7].

2. DERIVATION OF THE TOTAL PRECESSION IN A NEAR-CIRCULAR ORBIT

Schiff's formula for the geodetic precession of a gyroscope moving with velocity $\mathbf{v}$ at a distance $\mathbf{r}$ from a spherical earth (equation (1) of paper (A) in this series) may be rewritten in terms of the local gravitational acceleration $\mathbf{g}$ at the location of the gyroscope as follows:

$$\mathbf{\Omega}_G = \frac{3}{2c^2}(\mathbf{g} \times \mathbf{v}) \quad . \tag{1}$$

To calculate the total geodetic precession (*i.e.*, the main Schiff term plus the oblateness correction), we compute $\mathbf{g} \times \mathbf{v}$ for an actual orbit around the oblate earth.

Neglecting terms of second order in J_2 and the mean eccentricity e, the simplest description of a near-circular orbit around the oblate earth is as follows.

(1) The "mean position" describes a circle of radius $\bar{r}$ in a precessing plane with constant inclination i, the circle being described at a constant rate

$$\dot{\theta} = \sqrt{\frac{GM}{\bar{r}^3}} \left\{ 1 + J_2 \left(\frac{R_e}{\bar{r}}\right)^2 \left[\frac{9}{4} - \frac{21}{8}\sin^2 i\right] \right\} \quad , \tag{2}$$

and the precession rate about the North Pole being

$$\dot{\lambda}_A = \frac{-3}{2} J_2 \left(\frac{R_e}{\bar{r}}\right)^2 \dot{\theta} \cos i \quad , \tag{3}$$

where M is the mass and R_e the mean equatorial radius of the earth.

(2) The actual position is displaced from the mean position by amounts δr, $\delta\theta$ in the precessing plane only, as follows:

$$\delta r = \bar{r}\left\{\frac{1}{4}J_2\left(\frac{R_e}{\bar{r}}\right)^2 \sin^2 i \cos 2\theta - e\cos(\theta - \theta_p)\right\} \tag{4}$$

$$\delta\theta = \frac{1}{8}J_2\left(\frac{R_e}{\bar{r}}\right)^2 \sin^2 i \sin 2\theta + 2e\sin(\theta - \theta_p) \quad , \tag{5}$$

θ being measured from the equator, and θ_p being the phase angle defining the direction of perigee of the Keplerian ellipse on which the perturbations from J_2 are superimposed.

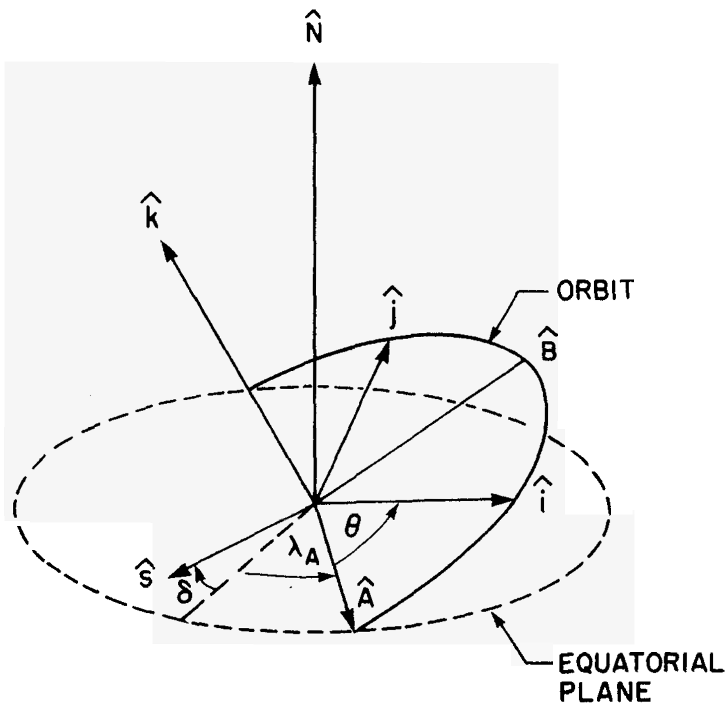

FIGURE 1. System of coordinates for a regressing orbit.

Adopting unit vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{\mathbf{k}}$ illustrated in figure 1, with $\hat{\imath}$ vertically upward, $\hat{\jmath}$ forward and $\hat{\mathbf{k}}$ perpendicular to the precessing plane, the actual position (to first order) is

$$\mathbf{r} = (\bar{r} + \delta r)\hat{\imath} + \bar{r}\delta\theta\,\hat{\jmath} \quad . \tag{6}$$

The angular velocity of the unit vector frame is

$$\omega_F = \dot{\theta}\hat{\mathbf{k}} + \dot{\lambda}_A \widehat{\mathbf{N}} \tag{7}$$

where $\widehat{\mathbf{N}} = \hat{\imath} \sin i \sin\theta + \hat{\jmath} \sin i \cos\theta + \hat{\mathbf{k}} \cos i$ is a unit vector directed northward along the earth's polar axis (see figure 1). The actual velocity relative to the earth's center is

$$\mathbf{v} = \delta\dot{r}\,\hat{\imath} + \bar{r}\delta\dot{\theta}\,\hat{\jmath} + \omega_F \times \mathbf{r} \quad , \tag{8}$$

which gives to first order

$$\begin{aligned} \mathbf{v} = (\delta\dot{r} - \bar{r}\dot{\theta}\delta\theta)\hat{\imath} \quad &+ \left\{\dot{\theta}\left[1 - \tfrac{3}{2}J_2\left(\tfrac{R_e}{\bar{r}}\right)^2 \cos^2 i\right](\bar{r} + \delta r) + \bar{r}\delta\dot{\theta}\right\}\hat{\jmath} \\ &+ \tfrac{3}{4}\left\{J_2\left(\tfrac{R_e}{\bar{r}}\right)^2 \bar{r}\dot{\theta} \sin 2i \cos\theta\right\}\hat{\mathbf{k}} \quad . \end{aligned} \tag{9}$$

The local gravity is

$$\mathbf{g} = -GM\frac{\mathbf{r}}{|\mathbf{r}|^3} + \nabla\left\{\frac{GMJ_2R_e^2}{|\mathbf{r}|^3}\left[\frac{1}{2} - \frac{3}{2}\frac{(\mathbf{r}\cdot\widehat{\mathbf{N}})^2}{|\mathbf{r}|^2}\right]\right\} \quad , \tag{10}$$

which gives to first order

$$\begin{aligned} \mathbf{g} = \quad & - \tfrac{GM}{\bar{r}^2}\left\{1 - 2\tfrac{\delta r}{\bar{r}} + \tfrac{3}{2}J_2\left(\tfrac{R_e}{\bar{r}}\right)^2\left[1 - \tfrac{3}{2}\sin^2 i(1 - \cos 2\theta)\right]\right\}\hat{\imath} \\ & - \tfrac{GM}{\bar{r}^2}\left\{\delta\theta + \tfrac{3}{2}J_2\left(\tfrac{R_e}{\bar{r}}\right)^2 \sin^2 i \sin 2\theta\right\}\hat{\jmath} \\ & - \tfrac{3}{2}\tfrac{GM}{\bar{r}^2}\left\{J_2\left(\tfrac{R_e}{\bar{r}}\right)^2 \sin 2i \sin\theta\right\}\hat{\mathbf{k}} \quad . \end{aligned} \tag{11}$$

It is easy to verify that equation (11) is, to first order, the time derivative $\dot{\mathbf{v}}$ of the velocity vector, and hence to justify the description of the orbit by the four equations (2) through (5) in (1) and (2). Neglecting terms of second order, we have from equation (9)

$$\begin{aligned} \dot{\mathbf{v}} \quad = \quad & (\delta\ddot{r} - \bar{r}\dot{\theta}\delta\dot{\theta})\hat{\imath} + (\dot{\theta}\delta\dot{r} + \bar{r}\delta\ddot{\theta})\hat{\jmath} \\ & - \tfrac{3}{4}\left\{\bar{r}\dot{\theta}^2 J_2\left(\tfrac{R_e}{\bar{r}}\right)^2 \sin 2i \sin\theta\right\}\hat{\mathbf{k}} + \omega_F \times \mathbf{v} \quad , \end{aligned} \tag{12}$$

and this yields, with appropriate substitution from the derivatives of equations (4) and (5), the expression for $\mathbf{g}$.

Equations (9) and (11) are the expressions for $\mathbf{v}$ and $\mathbf{g}$ that have to be substituted in equation (1) to get the total geodetic drift of the gyroscope including the effects of the earth's oblateness coefficient J_2 and the eccentricity e of the orbit. The result to first order is

$$\begin{aligned} \Omega_G &= \frac{3}{2c^2}\frac{(GM)^{3/2}}{\bar{r}^{5/2}} \\ &\left\{\hat{\mathbf{k}}\left[1 + J_2\left(\frac{R_e}{\bar{r}}\right)^2\left(\frac{9}{4} - \frac{27}{8}\sin^2 i + \frac{9}{4}\sin^2 i\cos 2\theta\right) + \frac{1}{\dot{\theta}}\delta\dot{\theta} - \frac{1}{\bar{r}}\delta r\right]\right. \\ &\left. - \frac{3}{4}J_2\left(\frac{R_e}{\bar{r}}\right)^2(2\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta)\sin 2i\right\} \end{aligned} \tag{13}$$

Now $2\hat{\imath}\sin\theta + \hat{\jmath}\cos\theta = \frac{3}{2}\hat{\mathbf{B}} - \frac{1}{2}\hat{\mathbf{B}}\cos 2\theta + \frac{1}{2}\hat{\mathbf{A}}\sin 2\theta$, where, as shown in figure 1, $\hat{\mathbf{A}}$ is the unit vector along the upward vertical at the ascending node and $\hat{\mathbf{B}}$ the unit vector along a direction 90° ahead of $\hat{\mathbf{A}}$ in the precessing plane.

Clearly the *averages per orbit* of $J_2\hat{\mathbf{B}}\cos 2\theta$, $J_2\hat{\mathbf{A}}\sin 2\theta$, $J_2\hat{\mathbf{k}}\cos 2\theta$, $\hat{\mathbf{k}}\delta\dot{\theta}$ and $\hat{\mathbf{k}}\delta r$ are of second order. To first order, this leaves as our final expression for Ω_G

$$\begin{aligned} (\Omega_G)_{AV} &= \frac{3}{2c^2}\frac{(GM)^{3/2}}{\bar{r}^{5/2}} \\ &\left\{\hat{\mathbf{k}}_{AV}\left[1 + J_2\left(\frac{R_e}{\bar{r}}\right)^2\left(\frac{9}{4} - \frac{27}{8}\sin^2 i\right)\right]\right. \\ &\left. - \frac{9}{8}\hat{\mathbf{B}}_{AV}J_2\left(\frac{R_e}{r}\right)^2\sin 2i\right\} \quad , \end{aligned} \tag{14}$$

where the expressions $\hat{\mathbf{k}}_{AV}$ and $\hat{\mathbf{B}}_{AV}$ serve to remind us that $\hat{\mathbf{k}}$ and $\hat{\mathbf{B}}$ each change by a small amount over the course of an orbit because of the nodal regression.

3. CONCLUDING OBSERVATIONS

It may seem strange that the eccentricity does not appear in equation (14). The reason is that the lowest order at which terms in e enter $(\Omega_G)_{AV}$ is the second, the formula for the average geodetic precession in an elliptic orbit around the nonoblate earth being just

$$(\Omega_G)_e = \frac{3}{2c^2}\frac{(GM)^{3/2}}{a^{5/2}(1-e^2)} \quad , \tag{15}$$

where a is the semimajor axis; so with an eccentricity of 10^{-3} the corrections are of order 6.9×10^{-3} milliarc-sec/yr, and hence negligible.

The corrections for J_2, on the other hand, are not negligible. In an equatorial orbit the correction is a precession in the plane of the orbit of magnitude $\frac{9}{4}J_2\left(\frac{R_e}{\bar{r}}\right)^2 A_G = +14$ milliarc-sec/yr. In a polar orbit it is a precession again in the plane of the orbit of magnitude -7 milliarc-sec/yr. In inclined orbits the corrections are a mixture of a linear term in the east-west plane and terms singly periodic in the regression rate in both the east-west and north-south planes. The question of separating the effects in various orbits has already been discussed in the first paper in this series.

Another point that may seem strange is that the numerical values for the oblateness correction given here differ from those given by Barker and O'Connell [4]. Both sets of values are correct; the discrepancy comes about from a difference in definition of the orbit parameters. Here the orbit is defined in terms of a *mean radius* and the *nodal period*, *i.e.*, the period between ascending nodes. Barker and O'Connell calculated precessions with respect to a radial distance not exactly equal to the mean radius and used a period not in general equal to the nodal period. The small difference in radius modifies the calculated value of the principal terms in the geodetic precession (*i.e.*, the terms for a spherical earth), and the correction for it needs to be taken into account in computing the total theoretical precession to be compared with experiment.

In highly elliptic orbits e is no longer comparable with J_2 and more complex terms make their appearance. Hoots and Fitzpatrick [5] have shown that components of Ω_G appear along $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{k}}$ proportional to $J_2e^2 \sin 2\theta_p$, to $J_2e^2 \cos 2\theta_p$, and to $(J_3/J_2)e$ times $\sin\theta_p$ and $\cos\theta_p$. Since the oblateness makes the perigee of the ellipse advance around the earth with a period which is about 15 weeks for a polar orbit, these effects will be modulated with periods different from those of any other terms, and can be determined absolutely in data analysis. However, our current view is that the disadvantages of going into a highly elliptic orbit outweigh the potential advantage of separating the terms by this method.

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