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CO-CO-EXPERIMENTS IN GRAVITATIONAL PHYSICS WITH GP-B AND STEP

M. B. Tapley and C. W. F. Everitt

GP-B Hansen Labs, Stanford University, Stanford, CA 94305-4085, U.S.A.

ABSTRACT

We show that the Gravity Probe B (GP-B) and the Satellite Test of Equivalence Principle (STEP) missions, with their disturbance-free orbits and precise tracking, will facilitate determination of at least two effects of post-Newtonian gravitation. First, for Einstein's classical test of general relativity, the advance of periapse of an orbit, the GP-B determination will be 3 parts in 1000. Second, for the eccentricity of objects orbiting the Earth as it orbits the sun, the GP-B determination will be 200 times more sensitive than lunar laser ranging measurements and at least 5 times more sensitive than Lageos due to GP-B's lower orbit. Nordtvedt shows that the annual variation in the eccentricity of such an orbit is zero only if general relativity is correct in its choice of parameters in the Parameterized Post-Newtonian generalized formulation of metric theories of gravitation. This test will discriminate between relativity and other theories at a level of 6 parts in 10^4. The perigee advance test also provides the most sensitive available test of the exotic β parameter associated in competing theories of gravitation with the second moment of the Earth's gravitational self-energy.

INTRODUCTION

The Gravity Probe B (GP-B) and Satellite Test of Equivalence Principle (STEP) satellites carry experiments in gravitational physics following carefully determined, drag-free orbits. The precisely determined orbits of the satellites can themselves serve the study of gravitational physics. The effects we wish to observe are small, and errors introduced by other sources have in the past been large enough to mask them. We found that observations of the GP-B and STEP orbits can make significant contributions to determinations of the Nordtvedt Effect and the advance of perigee.

THE NORDTVEDT EFFECT

The parameterized post-Newtonian (PPN) formalism is a generalized means of expressing metric theories of gravitation. K. Nordtvedt /1/ shows the importance of considering gravitomagnetic terms in the PPN formalism to account for observations of Earth satellite orbits. He demonstrates that the expression for the eccentricity induced in an Earth satellite's orbit by the gravitomagnetic effect of the sun includes the factor $(\Delta - 2\gamma - 2)$. In this expression, Δ is a parameter determining the size of the gravitomagnetic effect in the PPN framework, and γ is the parameter measured in photon-deflection and time-of-flight experiments. For general relativity, this factor is zero. Nordtvedt argues that the failure of observations to discern any induced eccentricity in Earth satellite orbits constitutes a successful null test of general relativity. Nordtvedt reduces the gravitomagnetic effect on an orbit to a Newtonian perturbation:

$$\delta\alpha = (\Delta-2\gamma)GM_e\frac{(\mathbf{w}\bullet\mathbf{v})\,\mathbf{r}}{c^2r^3} + GM_e\frac{(\mathbf{r}\bullet\mathbf{w})\mathbf{v}}{c^2r^3}, \qquad (1)$$

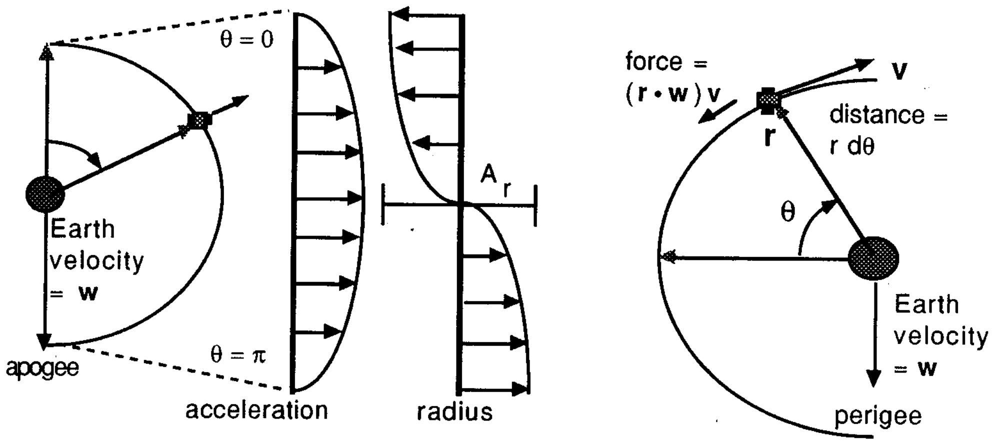
where G is the gravitational constant, M_e is the mass of the Earth, **w** is the velocity of the Earth in its solar orbit, v and **r** are the velocity and radius vector of the satellite relative to the Earth, c is the speed of light, and r is the scalar value of **r**. Both addends affect the satellite's eccentricity, but they act in opposite directions (Figure 1). The first raises one end of the orbit and lowers the other by exerting force radial to the Earth. We assume a linear satellite track and calculate its deflection to find the induced eccentricity. The other increases eccentricity by reducing orbit energy for half of the orbit and increasing it for the other half. (The resulting system is not conservative.) We calculate eccentricity change by computing energy loss or gain per half-orbit. We obtain the total orbit altitude variation per orbit by subtracting the effects of the two terms and multiplying by two:

$$\frac{A_r}{\text{orbit}} = 2\left[(\Delta-2\gamma)\frac{wvr}{c^2}\left(\frac{\pi}{2}\right) - \frac{wvr}{c^2}\pi\right] = (\Delta-2\gamma-2)\frac{wvr}{c^2}\pi \quad (2)$$

The size of the effect scales as 1/a for circular orbits of radius a. To find the maximum size of the altitude variation, we assume a stable situation in which each orbit contributes the above amount to the eccentricity vector, and the tip of the eccentricity vector describes a circle each year. In this case, the radius of the circle will be the maximum altitude variation. Using equation (2), we see that this radius is given by

$$A_{max} = \frac{\omega}{\Omega}\frac{A_r}{\text{orbit}}\frac{1}{2\pi} = (\Delta-2\gamma-2)\, GM_e \frac{wv}{r^2c^2 2\omega\Omega}, \quad (3)$$

where Ω is the radian frequency of the Earth's orbit and ω that of the satellite's orbit. If the satellite's orbit lies in the ecliptic, this agrees with Nordtvedt's result. If not, the dot products in equation (1) may be reduced. The dot products go as the sine of ϕ, the angle between the Earth's velocity vector and the perpendicular to the satellite's orbit. For GP-B, ϕ will vary from zero to 90° twice a year, becoming zero when the Earth is in line with Rigel and the sun.For STEP it will remain close to 90°, because of its sun-synchronous orbit.



a) radial force effect b) along-track force effect

Figure 1 Eccentricity changes due to radial acceleration and energy change

We compute the accuracy with which we determine (Δ - 2γ - 2) by dividing the accuracy of orbit determination by A_{max}. In Table 1, the orbit accuracy column describes the estimated cumulative uncertainty due to all orbital perturbations. The numbers given are considerably larger than the formal covariances produced by the methods described below, to reflect the influence of systematic or other unmodeled errors in a real-world solution. The lower values for STEP and GP-B indicate the use of more data and Global Positioning System (GPS) tracking to determine their orbits. The bodies listed here follow essentially drag-free orbits, so only gravitational perturbations will significantly contribute to errors. We did not consider non-drag-free satellites such as TOPEX.

TABLE 1 Accuracy of Determination of Nordtvedt Parameter

Earth satellite	orbit(cm)	Amax(cm)	(Δ - 2γ - 2)
Moon	10	86.5	0 ± 0.1113
Lageos	10	2604	0 ± 0.00384
GP-B	3	4726	0? ± 0.000635
STEP	3	4743	0? ± 0.000632

The unique nature of the resulting change in the orbit discriminates the Nordtvedt effect from other perturbations. For example, no term in the static geopotential field causes such a perturbation. To show the effect of the geopotential on the orbit, we quantify the eccentricity as a vector, (ξ,ψ), which points toward perigee and has a magnitude proportional to the eccentricity. ξ points to the ascending node of the satellite, while ψ is 90° behind ξ. Axelrad /2/ shows that considering only the effect of the Earth's oblateness (the J_2 term in a geopotential expansion), the ξ, ψ system is a simple harmonic oscillator, circulating about (ξ,ψ) = (0,0) with a period of 101 days for GP-B. The even zonal harmonics have no long-term effect on the eccentricity. However, the cumulative effect of the first 17 odd zonals (degrees 3 through 35) on the orbit of GP-B is to decrease ξ by $0.8940*10^{-6}$ per orbit. Combined with the J_2 eccentricity motion, the odd zonals provide an offset from $(\xi,\psi) = (0,0)$ to $(\xi,\psi) = (0, 0.0013)$ in the equilibrium point of the J_2 circulation. This new equilibrium point, the frozen eccentricity, depends on the value of J_2 and the lumped effect of the odd zonals. The tesseral terms in the geopotential field have no long-term effect (barring deep resonance, in which case the time signature will differ from the Nordtvedt effect). Therefore, the Nordtvedt effect will be easily distinguishable from geopotential effects. The tidal potential does contain some terms with annual periods, but a tidal constituent with frequency ω causes eccentricity perturbations at frequencies $(\omega + \Omega')$ and $(\omega - \Omega')$, where Ω' is the satellite's J_2-induced rate of perigee advance /3/. GP-B has an Ω' about 3.5°/day; no terms in the tidal spectrum have periods of a year plus or minus this amount. Another way to discriminate between the tidal effect and the Nordtvedt effect is the direction in which the eccentricity shifts. The Nordtvedt effect causes eccentricity shift in the direction of Earth's motion. The tidal effect may be in other directions and differs for different orbits. Also, the effect for GP-B is modulated by the angle ϕ, whereas it is nearly constant for STEP.

PERIGEE PRECESSION

Overview

The 0.43 arcsecond per year precession in Mercury's orbit is a classical test of general relativity. The precessions in the orbits of STEP and GP-B are smaller, but their orbits are better observed. Because of uncertainties in the sun's mass distribution, the perigee precession determinations for these two satellites will be comparable to those for Mercury. According to Misner, Thorne, and Wheeler /4/, the expression which gives the advance in perigee per orbit is

$$\delta \phi = \frac{2 \Pi}{\sqrt{1 - \frac{M}{r_0}}} , \qquad (4)$$

where $\delta\phi$ is the angle from one perigee to the next, M is the mass of the Earth, and r_0 is the radius of the orbit.

TABLE 2 Relativistic and J_2 Perigee Advances for Earth Satellites in Radians per Second

Earth satellite	Relativity perigee advance	J_2 perigee advance
Moon	$-9.2*10^{-17}$	$-1.9*10^{-12}$
Lageos	$-4.5*10^{-13}$	$3.7*10^{-8}$
GP-B	$-2.03*10^{-12}$	$7.16*10^{-7}$
STEP	$-2.05*10^{-12}$	$6.73*10^{-7}$

The current uncertainty in J_2 is about 2 parts in 10^7. From ascending node rate and short-term measurements, the GP-B and STEP missions will reduce that to 3 parts in 10^{10}. Uncertainty in the odd harmonics will change only the equilibrium point of the eccentricity circulation, not the period. Therefore, the overall uncertainty in perigee precession from geopotential effects is 1 part in 10^5 of the relativistic precession.

Tidal perturbations might also affect the rate of perigee precession if a tide-induced precession of the perigee is sufficiently close to the 101-day period of the J_2 precession. Casotto /5/ indicates that there is no such tidal effect on the eccentricity. No term in the tidal spectrum has a period of 101 days, so tidal effects will not mask the relativistic perigee precession.

Numerical study for laser measurements

The accuracy of determination of perigee location limits the measurement of perigee precession. To assess this accuracy, we use a numerical approximation suggested by J.V. Breakwell /6/. The objective is to estimate the accuracy of determination of rate of perigee advance of GP-B's orbit given a set of measurements from laser ranging stations around the Earth. We recognize that there is no way to measure perigee directly, especially in a situation such as this where the orbit is nearly circular. We can achieve only a statistical determination based on incomplete measurements applied to adjusting parameters in a model of the orbit. We assume that the measurement set GP-B or STEP returns is sufficient to make this determination. Our success in the approximations gives reassurance in this. We also assume that long-term trends in the modelled argument of perigee will accurately represent long-term trends in the "actual" argument of perigee. The results we present below describe formal uncertainties modified to represent real-world errors which will inevitably arise. For reasons of simplicity, this study ignores GPS data, though GP-B and STEP will both carry GPS receivers. The accuracy of orbit determination using GPS data will be substantially better than for laser data alone, so results of this study are somewhat conservative.

The overall method of the estimate is to choose a set of parameters describing the orbit, the Earth, and the tracking system. We then produce the covariance matrix $P = (H^T R^{-1} H)^{-1}$ resulting from a least-squares estimate of the parameters using laser range data from the stations. Among the orbit parameters are the frozen value of the eccentricity (ψ_F) the instantaneous eccentricity vector (ξ_0, ψ_0), and two gravitational parameters: the rate of orbit ascending node advance (Ω'), and the rate of perigee advance (ω'). The significant results appear in Table 3. The uncertainty in perigee advance rate is 0.0393 arcsec/yr, or 0.00297 times the relativistic perigee advance rate. Thus the numerical approximation indicates that we will be able to test relativity to three parts in 1000.

TABLE 3 Results of Calculations

parameter	analysis uncertainty	GEODYN uncertainty
"frozen" eccentricity (ψ_F)	$1.719*10^{-12}$	$2*10^{-12}$
equatorial eccentricity (ξ_0)	$1.956*10^{-12}$	$2*10^{-12}$
polar eccentricity (η_0)	$2.023*10^{-12}$	$2*10^{-12}$
ascending node advance rate (Ω')	$9.3*10^{-6}$ arcsec/yr	$2.8*10^{-5}$ arcsec/yr
perigee advance rate (ω')	0.0393 arcsec/yr	0.036 arcsec/yr

Computer simulation for GPS measurements

To overcome limitations in the numerical approximation, we perform a simulation of the orbit determination process using the GEODYN program. Again, only laser ranging data are included. As with the data reduction from the actual mission, we use a long-arc solution. We set uncertainty in earth geopotential coefficients and in tidal perturbations equal to their values in current Earth models. GEODYN can not solve for the same variables that the approximation used, so we make the following substitutions: instead of solving for frozen and circulating eccentricity separately, we solve for the two components of the eccentricity vector. To get node rate accuracy and perigee rate accuracy, we take the accuracy of static angle determination for a 101-day arc, multiply by two, and divide by a year. Formal uncertainties produced by GEODYN are multiplied by two to account for the effects of mis-modelling, process noise and other unmodeled problems. We include this correction in Table 3. The simulation confirms the numerical approximation's results.

CONCLUSION

The theory of relativistic advance of perigee is well known but not well measured. Current estimates of the uncertainty in its determination through observations of Mercury's orbit are about 0.3%. In the GP-B / Earth system, the precession will be smaller, but the observations will be more accurate. No geopotential or tidal effect except J_2 will cause perigee precession with a similar signature to the relativistic effect, and we will know J_2 to extremely high accuracy. The limiting factor on determination of the relativistic effect will thus be our ability to determine the location of perigee. We have shown that with reasonable assumptions, we can match the current accuracy of determination using lasers only. Improved accuracy due to GPS measurements could lower the uncertainty even more. In addition, we can provide a substantially better measurement of the Nordtvedt effect than is currently possible. This will serve to constrain the combination of frame-dragging and light deflection PPN parameters ($\Delta - 2\gamma - 2$) to 0 ± 0.0006. In addition, because the GP-B and STEP orbits are relatively close to the Earth's surface in comparison to their orbit radii, perigee advance measurements on them will provide the best test anywhere /7/ of the exotic β parameter associated with the second moment of the Earth's gravitational self-energy.

References:

1. K. Nordtvedt, Gravitomagnetic interaction & laser ranging earth satellites, *Phys. Rev Lett.*, 61(23), 2647 (1988).

2. P. Axelrad, *A Closed Loop GPS-Based Orbit Trim System for Gravity Probe B*, Stanford Univ., Hansen Laboratory, Stanford, 1990.

3. R. Eanes, private communication (1991).

4. C. Misner, K. Thorne, & J. Wheeler, *Gravitation*, Freeman & Company, New York, 1973.

5. S. Casotto, *Nominal Ocean Tide Models for TOPEX Precise Orbit Determination*, U. of Texas at Austin (1989).

6. J. Breakwell, private communication (1988).

7. K.Nordtvedt, private communication (1992).

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