MOTION OF A GYROSCOPE ACCORDING TO EINSTEIN'S THEORY OF GRAVITATION*

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The Experimental Basis of Einstein's Theory.—Einstein's theory of gravitation, the general theory of relativity, has been accepted as the most satisfactory description of gravitational phenomena for more than forty years. It is a theory of great conceptual and structural elegance, and it is designed so that it automatically agrees in the appropriate limits with Galileo's observation of the equality of gravitational and inertial mass, with Newton's mechanics of gravitating bodies, and with Einstein's special theory of relativity. Leaving aside the very important matter of elegance, we wish in this section to examine the experimental basis of the theory. This basis consists of the three points of limiting agreement with earlier results just mentioned, together with certain astronomical evidence.

The equality of gravitational and inertial mass was originally formulated in terms of equal accelerations for all freely falling test particles, regardless of mass or chemical composition. In this form, it is a consequence of general relativity theory insofar as test particles move in accordance with the geodesic equations for a Riemannian metric. However, the experimental evidence on freely falling particles is not of very great accuracy. Much more precise experiments were performed about half a century ago by Eötvös and collaborators,[1] and are now being repeated with improved technique by Dicke.[2] Since they make use of particles that are not in free fall but are subjected to nongravitational constraints, the relation with Einstein's theory is not quite so simple as just indicated. On the other hand, we can regard these experiments as establishing with great confidence the *principle of equivalence*, which we express in the following way: all observations made locally on a system in a static, uniform gravitational field in the absence of local background matter agree with corresponding observations made on the same system when it is subjected to an equivalent acceleration in the absence of the field.

This statement of the equivalence principle goes beyond the direct evidence of the Eötvös experiments. For one thing, the Eötvös experiments do not compare observations made in the presence and absence of a gravitational field, but rather compare observations made with an acceleration in one direction and a gravitational field in another. More important, the observations made are not perfectly general, but consist of mass comparisons. However, there is a great deal of physical content to a precise mass measurement, since many of the phenomena known in physics enter into it with sufficient effect to be noticeable;[3-5] this occurs through the Hamiltonian of the system, of which the mass is essentially the ground-state eigenvalue. It would be remarkable if the equivalence principle were to apply to the ground states of the Hamiltonians of physics, and not also to the excited states that determine, for example, the transition frequencies. Thus, while this formulation of the equivalence principle is an extrapolation from the direct evidence of the Eötvös experiments, it is not so great an extrapolation as might at first be supposed.

The other points of contact with Einstein's theory of gravitation are most readily discussed in terms of the formalism of the theory. In the simplest interesting case, that of the gravitational field about a stationary, spherically symmetric mass m, the metric that represents the solution of the field equations can be written in the Schwarzschild standard form[6]

$$ds^2 = (1 - 2m/r)dt^2 - dr^2/(1 - 2m/r) - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (1)$$

where units have been chosen so that the speed of light and the Newtonian gravitational constant are equal to one. Then special relativity is valid whenever m/r may be neglected in comparison with unity. The metric (1) may be supplemented by the geodesic equation of motion for a test particle, and the null-geodesic equation of motion for a light ray; alternatively, these equations of motion can be obtained from the field equations themselves.

Suppose now that we wish to verify the structure of equation (1) by comparison with observation. We may then write the metric in the form

$$ds^2 = (1 + \alpha m/r + \beta m^2/r^2 + \ldots)dt^2 - (1 + \gamma m/r + \delta m^2/r^2 + \ldots)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad (2)$$

where α, β, γ, δ, ... are of order unity. This implies an expansion in powers of the quantity m/r, which is very small in all cases of observational interest. It also implies spherical symmetry, in which case $d\theta$ and $d\phi$ appear in the combination shown and any multiplying series in powers of m/r is readily transformed away by a change of the radial variable. It then follows that the limiting case of Newtonian mechanics requires only that $\alpha = -2$. The gravitational red shift is also accounted for in the same way. The theory of the gravitational deflection of light passing close to the sun results from the null-geodesic equation for a light ray, together with the above value for α and the choice $\gamma = +2$. Finally, the theory of the precession of the perihelion of the orbit of the planet Mercury results from the geodesic equation of motion for a test particle, the foregoing values for α and γ, and the choice $\beta = 0$. Higher terms in the series of (2) have not been subjected to experimental test.[7]

We now ask to what extent the above numerical values of α, β, and γ, and the equations of motion, may be inferred without recourse to Einstein's theory of gravitation. It is argued elsewhere[8] that the values of α and γ, and the null-geodesic equation for a light ray, can be obtained correctly from the equivalence principle as formulated above, together with the special theory of relativity. On the other hand, the value of β, which depends in an essential way on the nonlinearity of the field equations, and the geodesic equation for a test particle, cannot be obtained in this way. Thus, the planetary orbit precession remains as the sole experimental basis for Einstein's theory. Recent terrestrial experiments on the gravitational red shift[9, 10] should be thought of as providing additional evidence for the equivalence principle as formulated above, rather than for the general theory of relativity.

There are at least three general ways in which one might look for new experimental verifications of Einstein's theory of gravitation to supplement the planetary orbit precession. The first is related to cosmological implications, such as variations of certain natural constants with time,[2] and the structure and evolution of the universe.[11] As an example of the latter, a search could be made for a minimum apparent size of extra-galactic nebulae, since the most distant visible nebulae might be expected to appear larger and redder than those at intermediate distances. The second approach consists in searching for gravitational radiation, either arising from extra-terrestrial sources, or possibly generated in the laboratory.[12] The third proposal is specifically designed to involve terms in the metric (2) beyond α and γ, and the equation of motion of matter of finite rest mass beyond the Newtonian approximation. This could be accomplished by measuring the precession of the spin axis of a torque-free gyroscope in the gravitational field of the rotating earth;[13, 14] although the β, δ, . . . terms in (2) are not involved, new off-diagonal space-time components of the metric tensor that arise from earth rotation become significant. The present paper is devoted mainly to a derivation and discussion of the results already published in a brief notice.[14]

Equations of Motion of a Spinning Test Particle.—Papapetrou[15] has derived covariant equations of motion for the center of mass and the spin angular momentum of a spinning test particle (torque-free gyroscope). These equations are incomplete in two respects. (1) They refer to free fall, that is, to motion in a pure gravitational field. Thus, while they describe a gyroscopic satellite, they do not describe a gyroscope in an earth-bound laboratory. Our first task, then, will be to generalize these equations by including a non-gravitational constraining force **F**. (2) There are only three independent equations for the six components of the angular momentum tensor, so that supplementary conditions must be imposed. These conditions may be chosen in either of two natural ways. Corinaldesi and Papapetrou[16] require the timelike components of the angular momentum tensor to vanish in the coordinate system in which the central attracting body is at rest. Pirani[17] requires these components to vanish in the rest-frame of the gyroscope. We shall consider both of these possibilities here.

Papapetrou uses the method of Fock[18] to obtain the equations of motion. This consists in starting with the "dynamical equation" for the energy-momentum tensor, according to which its covariant divergence is zero:

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{3}$$

$T^{\mu\nu}$ represents the test particle, and is supposed to vanish outside of a narrow tube in four-dimensional space-time that surrounds the world line of some representative point X^μ of the particle. The space components of X^μ, X^i ($i = 1, 2, 3$), are regarded as functions of the time $X^4 = t$, or of the proper time s along the world line. This world line need not be a geodesic of the gravitational field, which is described by the metric tensor $g_{\mu\nu}$. The equations of motion are obtained by manipulating integrals of the form

$$\int T^{\mu\nu}\, dv, \qquad \int (x^\alpha - X^\alpha)T^{\mu\nu} dv, \qquad \ldots, \tag{4}$$

where the integrations extend over three-dimensional space for t = constant. A point test particle is one for which some of the components of the first integral of (4) fail to vanish, but the other integrals are always zero. A spinning test particle is one for which some of the components of both integrals fail to vanish, but an integral containing more than one factor $(x^\alpha - X^\alpha)$ is always zero.

The four-velocity of the representative point of the particle is $u^\mu = dX^\mu/ds$. We generalize the nongravitational constraining force **F** to a four-force F^μ such that $F^i = \mathbf{F}$ and $F^\mu u_\mu = 0$,[19] and assume that it is applied at the point X^μ. Equation (3) is still valid, but $T^{\mu\nu}$ must now be thought of as consisting of two parts, one representing the test particle and the other the force. We avoid a detailed specification of the second part by assuming that its covariant divergence is zero except at the point of application, and is there proportional to F^μ. Then if we denote just the first part by $T^{\mu\nu}$, equation (3) is to be replaced by

$$T^{\mu\nu}{}_{;\nu} = (F^\mu/u^4)\delta(x^1 - X^1)\delta(x^2 - X^2)\delta(x^3 - X^3), \tag{5}$$

where the right side is a product of Dirac δ functions. That equation (5) is actually covariant is most readily shown by using it to calculate the equation of motion of a point test particle after the manner of Papapetrou. The result is

$$m_0(du^\mu/ds + \Gamma^\mu{}_{\alpha\beta}u^\alpha u^\beta) = F^\mu, \tag{6}$$

which reduces to the geodesic equation when $F^\mu = 0$. The rest mass of the particle,

$$m_0 \equiv (1/u^4)\int T^{44}\, dv, \tag{7}$$

is a scalar,[15] and can be shown to be constant along the world line.

We may also apply Papapetrou's procedure to a spinning test particle, using, however, equation (5) in place of (3). The spin angular momentum is defined by

$$S^{\mu\nu} \equiv \int (x^\mu - X^\mu)T^{\nu 4}\, dv - \int (x^\nu - X^\nu)T^{\mu 4}\, dv, \tag{8}$$

which can be shown explicitly to have the transformation properties of a tensor.[15] The equation of motion of the spin is unaffected by inclusion of F^μ, and may be written in the covariant form[15]

$$(DS^{\mu\nu}/Ds) + u^\mu u_\alpha(DS^{\nu\alpha}/Ds) - u^\nu u_\alpha(DS^{\mu\alpha}/Ds) = 0, \tag{9}$$

where D/Ds represents covariant differentiation along the world line:

$$(DS^{\mu\nu}/Ds) \equiv (dS^{\mu\nu}/ds) + \Gamma^\mu{}_{\alpha\beta}S^{\alpha\nu}u^\beta + \Gamma^\nu{}_{\alpha\beta}S^{\mu\alpha}u^\beta. \tag{10}$$

The equation of motion of the representative point of the particle is modified from equations (6) and (7) by terms of order $S^{\mu\nu}$. However, this effect of the spin on the orbital motion of the particle is completely negligible in situations of current experimental interest.

It is sometimes convenient to rewrite equation (9) in the noncovariant form

$$(DS^{\mu\nu}/Ds) + (u^\mu/u^4)(DS^{\nu 4}/Ds) - (u^\nu/u^4)(DS^{\mu 4}/Ds) = 0. \tag{11}$$

This may be obtained from equation (9) in the following way:[20] set $\nu = 4$ in (9) and multiply by (u^ν/u^4); then set $u = 4$ in (9) and multiply by (u^μ/u^4); then add these two equations and substitute into (9). Since (11) is antisymmetric in u and ν, it seems at first to comprise six independent equations. However, if we put $\mu = i$, $\nu = 4$, we obtain a trivial identity. Thus only three of these equations are actually independent; the same remark applies to (9). It is therefore necessary to impose a supplementary condition.

The physical meaning of the supplementary condition is best seen by writing $S^{\mu\nu}$ in rectangular coordinates. Then since T^{i4} is the momentum density in the i direction, it follows from equation (8) that

$$\mathbf{S} = (S_x, S_y, S_z) = (S^{23}, S^{31}, S^{12}) \tag{12}$$

is the spin angular momentum vector with respect to the representative point

$$\mathbf{r} = (X^1, X^2, X^3). \tag{13}$$

Further, for $\mu = i$, $\nu = 4$, the second integral in (8) is zero since the integration extends over three-dimensional space for constant time, so that x^4 and X^4 are both equal to t. We thus obtain

$$S^{i4} = \int (x^i - X^i)T^{44}\, dv = m_0 u^4 \epsilon^i, \tag{14}$$

where m_0 is given by (7), and ϵ^i is the position of the center of mass of the particle with respect to X^i.

The Corinaldesi-Papapetrou (CP) supplementary condition is[16]

$$S^{i4} = 0 \tag{15}$$

in the rest-frame of the central attracting body, so that X^i is the center of mass in this coordinate system. On the other hand, Pirani's supplementary condition is[17]

$$S^{\mu\nu}u_\nu = 0. \tag{16}$$

In the rest-frame of the particle, $u^i = 0$ and u_i is zero or negligibly small,[21] so that $S^{i4} = 0$ and X^i is the center of mass in the particle rest-frame. As is well known,[22] the position of the center of mass of an object that possesses internal angular momentum is not a Lorentz-invariant quantity. It follows that the supplementary condition removes the ambiguity inherent in the choice of the representative point X^i in terms of which the motion of the spinning test particle is described and at which the constraining force is applied, by specifying that this point is the center of mass in one or another coordinate system.

In using the CP condition, it is more convenient to start from equation (11) than from (9). With the help of (15) and (10), equation (11) becomes

$$(DS^{ij}/Ds) = \Gamma^4{}_{k\mu}(u^\mu/u^4)\,(S^{ik}u^j - S^{jk}u^i). \tag{17}$$

We now rewrite (17) in terms of the rectangular coordinates (12) and (13), making use of the standard form (1) of the Schwarzschild metric. To lowest order, which requires knowledge only of α and γ in (2), we obtain[16]

$$(d\mathbf{S}/dt) = (m/r^3)[2\mathbf{S}(\mathbf{r}\cdot\mathbf{v}) + 2\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{S}) - (3\mathbf{r}/r^2)(\mathbf{r}\cdot\mathbf{v})(\mathbf{r}\cdot\mathbf{S})], \tag{18}$$

where $\mathbf{v} = d\mathbf{r}/dt$. It is instructive also to find the analog of (18) when the isotropic form of the Schwarzschild metric is used;[6] to first order, this metric is

$$ds^2 = (1 - 2m/r)dt^2 - (1 + 2m/r)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2), \tag{19}$$

and is the same to this order as the metric expressed in Fock's harmonic coordinates.[23] Equation (17) then becomes to lowest order

$$(d\mathbf{S}/dt) = (m/r^3)[3\mathbf{S}(\mathbf{r}\cdot\mathbf{v}) + \mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - 2\mathbf{r}(\mathbf{v}\cdot\mathbf{S})]. \tag{20}$$

In using the Pirani condition (16), it is more convenient to start from equation (9) than from (11). We differentiate (16) and substitute into (9) to obtain

$$(DS^{\mu\nu}/Ds) = (u^\mu S^{\nu\alpha} - u^\nu S^{\mu\alpha})(du_\alpha/ds - u^\beta u_\rho \Gamma^\rho{}_{\alpha\beta}). \tag{21}$$

Since $u_\alpha = g_{\alpha\gamma}u^\gamma$, du_α/ds can be expressed in terms of the nongravitational constraining acceleration $f^{\mu} = F^{\mu}/m_0$ by means of equation (6). In doing this, terms of higher than first order in $S^{\mu\nu}$ on the right side of (21) are neglected. After some reduction, (21) becomes to this approximation

$$(DS^{\mu\nu}/Ds) = (u^{\mu}S^{\nu\alpha} - u^{\nu}S^{\mu\alpha})f_{\alpha}. \tag{22}$$

The components S^{i4} and f_4 may be eliminated from (22) by using the supplementary condition (16) and the relation $f_{\mu}u^{\mu} = 0$. The standard form of the Schwarzschild metric then gives for the spin equation of motion

$$(d\mathbf{S}/dt) = (3m/r^3)[\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - (\mathbf{r}/r^2)(\mathbf{r}\cdot\mathbf{v})(\mathbf{r}\cdot\mathbf{S})] + \mathbf{S}(\mathbf{v}\cdot\mathbf{f}) - \mathbf{f}(\mathbf{v}\cdot\mathbf{S}); \tag{23}$$

the isotropic form leads to

$$(d\mathbf{S}/dt) = (m/r^3)[\mathbf{S}(\mathbf{r}\cdot\mathbf{v}) + 2\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{S})] + \mathbf{S}(\mathbf{v}\cdot\mathbf{f}) - \mathbf{f}(\mathbf{v}\cdot\mathbf{S}). \tag{24}$$

Transformation to the Gyroscope Rest-Frame.—The four equations of motion—(18), (20), (23), and (24)—are quite different from each other, and yet presumably all describe the same physical system. They may be reconciled by transforming each to the rest-frame of the gyroscope. Physically, this corresponds to the fact that measurements on the gyroscope are most readily interpreted as being made by a co-moving observer, who may then transmit these measurements to the outside world. If, for example, the gyroscope is in a satellite, it may be observed by a human being who travels with it, or by a device which telemeters information to the earth below. One of the questions that is most important from an observational point of view is whether or not there is a change in the frequency of rotation of the gyroscope, since this would be the simplest quantity to measure with precision. Corinaldesi and Papapetrou[16] argued on the basis of equation (18), which is the only one of the four that they obtained, that since $(d\mathbf{S}/dt)$ has a component parallel to $\mathbf{S}$, its magnitude will change, and hence the rotation frequency might be expected to change. On the other hand, the four equations predict quite different values for the change in magnitude of $\mathbf{S}$.

The physical situation is as follows. The co-moving observer (human or otherwise) measures the rotation frequency by comparison with a standard clock that is carried along, and interprets this as being the ratio of the magnitude of the angular momentum to the moment of inertia, both measured in the gyroscope rest-frame. The moment of inertia is determined by the dimensions of the gyroscope, and hence by comparison with a standard measuring rod that is also carried along by the observer. Thus, if we transform $\mathbf{S}$ to the co-moving system, we have a consistent set of observations. As we shall see, it turns out that the magnitude of the transformed angular momentum $\mathbf{S}_0$ is constant. This means that if the dimensions do not change, the rotation frequency is constant, and the gyroscope behaves like a clock which can be set to any desired frequency. The frequency will of course exhibit Doppler and gravitational shifts when observed from the outside, just like that of a more conventional clock. This will occur, for example, if the gyroscope is in a satellite and its rotation frequency is telemetered to the earth below.

The transformation to the gyroscope rest-frame consists of two parts, a coordinate transformation involving changes of order m/r, and a Lorentz transformation involving changes of order v^2. Since we need work only to first order in m/r

and in v^2, we can deal with the two transformations separately, and combine their effects later.

The first transformation recognizes the fact that a coordinate length δx^i in the original system corresponds to a proper length $\delta s = (-g_{ii})^{1/2}\delta x^i$, and that it is this δs that is measured by the standard rod of the co-moving observer. Thus when the isotropic metric (19) is used, all coordinate lengths are to be multiplied by $(1 + m/r)$, to first order, to convert them to the co-moving system. From (12), we see that each component of $\mathbf{S}$ transforms like the product of two coordinates, so that $\mathbf{S}$ must be multiplied by $(1 + 2m/r)$ to obtain $\mathbf{S}_0$, again to first order:

$$\mathbf{S}_0 = (1 + 2m/r)\mathbf{S}. \tag{25}$$

With the standard metric (1) the situation is slightly more complicated. Here, radial length intervals are to be multiplied by $(1 + m/r)$, and tangential length intervals are to be left unchanged. It then follows from (12) that the tangential components of $\mathbf{S}$ are to be multiplied by $(1 + m/r)$, and the radial component is to be left unchanged:

$$\mathbf{S}_0 = \mathbf{S} + (m/r)[\mathbf{S} - (\mathbf{r}/r^2)(\mathbf{r}\cdot\mathbf{S})]. \tag{26}$$

The Lorentz transformation is most conveniently written down if $\mathbf{v}$ is chosen to be along one of the rectangular axes, say x:

$$\begin{aligned} S_0^{12} &= \gamma S^{12} + v\gamma S^{24}, & S_0^{23} &= S^{23}, & S_0^{31} &= \gamma S^{31} - v\gamma S^{34}, \\ S_0^{14} &= S^{14}, & S_0^{24} &= \gamma S^{24} + v\gamma S^{12}, & S_0^{34} &= \gamma S^{34} - v\gamma S^{31}; \end{aligned} \tag{27}$$

here, $\gamma = (1 - v^2)^{-1/2}$. The CP condition (15) requires that $S^{14} = S^{24} = S^{34} = 0$, so that equations (27) become, with the help of (12):

$$\begin{aligned} S_{0x} &= S_x, & S_{0y} &= \gamma S_y, & S_{0z} &= \gamma S_z, \\ S_0^{14} &= 0, & S_0^{24} &= v\gamma S_z, & S_0^{34} &= -v\gamma S_y. \end{aligned} \tag{28}$$

Thus, in this case, the component of $\mathbf{S}$ parallel to $\mathbf{v}$ is left unchanged, but the perpendicular components are multiplied by $(1 + {}^1/_2 v^2)$, to first order. This may be written in a rotation-covariant manner as follows:

$$\mathbf{S}_0 = \mathbf{S} + {}^1/_2[v^2\mathbf{S} - \mathbf{v}(\mathbf{v}\cdot\mathbf{S})]. \tag{29}$$

The Pirani condition (16) requires, when $\mathbf{v}$ is along x, that

$$S^{14} = 0, \qquad S^{24} = vS^{12}, \qquad S^{34} = -vS^{31}.$$

Here, effects of order m/r have been neglected, since we are permitted to separate the coordinate and Lorentz transformations in lowest order. Equations (27) then become

$$\begin{aligned} S_{0x} &= S_x, & S_{0y} &= \gamma^{-1}S_y, & S_{0z} &= \gamma^{-1}S_z, \\ S_0^{14} &= 0, & S_0^{24} &= 0, & S_0^{34} &= 0. \end{aligned} \tag{30}$$

The last three of equations (30) are in agreement with the remark made just after equation (16), that when the Pirani condition is used, $S^{i4} = 0$ and hence X^i is the center of mass in the gyroscope rest-frame. The first three of equations (30) show that the component of $\mathbf{S}$ parallel to $\mathbf{v}$ is left unchanged, while the perpendicular components are multiplied by $(1 - {}^1/_2 v^2)$, to first order. In rotation-covariant form this becomes

$$\mathbf{S}_0 = \mathbf{S} - {}^1/_2[v^2\mathbf{S} - \mathbf{v}(\mathbf{v}\cdot\mathbf{S})]. \tag{31}$$

Equation (18) was obtained by using the standard form of the Schwarzschild metric together with the CP supplementary condition. Thus, in order to transform it to the gyroscope rest-frame we combine the transformations (26) and (29) to first order:

$$\mathbf{S}_0 = \mathbf{S} + (m/r)[\mathbf{S} - (\mathbf{r}/r^2)(\mathbf{r}\cdot\mathbf{S})] + {}^1/_2[v^2\mathbf{S} - \mathbf{v}(\mathbf{v}\cdot\mathbf{S})]. \tag{32}$$

The time derivative of (32) is

$$(d\mathbf{S}_0/dt) = (d\mathbf{S}/dt) - (m\dot{r}/r^2)\mathbf{S} + (3mr/\dot{r}^4)\mathbf{r}(\mathbf{r}\cdot\mathbf{S}) - (m/r^3)\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - \\ (m/r^3)\mathbf{r}(\mathbf{v}\cdot\mathbf{S}) + v\dot{v}\mathbf{S} - {}^1/_2\dot{\mathbf{v}}(\mathbf{v}\cdot\mathbf{S}) - {}^1/_2\mathbf{v}(\dot{\mathbf{v}}\cdot\mathbf{S}). \tag{33}$$

Here, dots denote time derivatives, and terms of order $(m/r)(d\mathbf{S}/dt)$ and $v^2(d\mathbf{S}/dt)$ have been neglected in comparison with $(d\mathbf{S}/dt)$. Also, the difference between the differential time intervals dt in the two coordinate systems has been neglected, since it is of fractional order m/r and v^2. Several substitutions can be made on the right side of equation (33). It is evident that $\dot{r} = (\mathbf{r}\cdot\mathbf{v})/r$ and $\dot{v} = (\mathbf{v}\cdot\dot{\mathbf{v}})/v$. The acceleration $\dot{\mathbf{v}}$ can be obtained from the equation of motion (6). Since we require $\dot{\mathbf{v}}$ only to first order in m/r and v^2, spin corrections to (6) can be neglected and the Newtonian approximation used:

$$\dot{\mathbf{v}} = -(m/r^3)\mathbf{r} + \mathbf{f}, \tag{34}$$

where $\mathbf{f} = \mathbf{F}/m_0$ is the acceleration that arises from the nongravitational constraint. Finally, $(d\mathbf{S}/dt)$ is to be taken from (18). With all these substitutions, equation (33) becomes

$$(d\mathbf{S}_0/dt) = (3m/2r^3)[\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{S})] + \mathbf{S}(\mathbf{v}\cdot\mathbf{f}) - {}^1/_2\mathbf{f}(\mathbf{v}\cdot\mathbf{S}) - {}^1/_2\mathbf{v}(\mathbf{f}\cdot\mathbf{S}). \tag{35}$$

Since according to (32), $\mathbf{S}$ differs from $\mathbf{S}_0$ only by terms of order m/r and v^2, it may be replaced by $\mathbf{S}_0$ on the right side of (35).

Equation (20), which arises from the isotropic metric and the CP condition, may be transformed in just the same way. The transformation is the combination of (25) and (29):

$$\mathbf{S}_0 = \mathbf{S} + (2m/r)\mathbf{S} + {}^1/_2[v^2\mathbf{S} - \mathbf{v}(\mathbf{v}\cdot\mathbf{S})]. \tag{36}$$

Differentiation of (36), followed by substitution from (20) and (34), again leads to equation (35).

When the Pirani condition is used, equation (23) with the standard metric must be transformed by means of (26) and (31), and equation (24) with the isotropic metric must be transformed by means of (25) and (31). In both cases, the result analogous to equation (35) is

$$(d\mathbf{S}_0/dt) = (3m/2r^3)[\mathbf{v}(\mathbf{r}\cdot\mathbf{S}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{S})] + {}^1/_2\mathbf{v}(\mathbf{f}\cdot\mathbf{S}) - {}^1/_2\mathbf{f}(\mathbf{v}\cdot\mathbf{S}). \tag{37}$$

Again, $\mathbf{S}$ may be replaced by $\mathbf{S}_0$ on the right side.

The difference between equations (35) and (37) evidently arises from the difference between the CP and Pirani supplementary conditions rather than from the difference between the standard and isotropic forms of the metric. It is therefore related to the specification of the representative point (13), at which $\mathbf{F}$ is applied, as being the center of mass in one or another coordinate system. The specification of this point has no physical significance so long as $\mathbf{F} = 0$, and indeed we note that equations (35) and (37) agree in this case. However, when $\mathbf{F} \neq 0$, it will result in a torque $-\epsilon \times \mathbf{F}$ about the center of mass, where ϵ is the vector position of the center of mass with respect to the representative point. Since measurements are to be made by a co-moving observer, such a torque in his coordinate system is instrumental, and to be avoided by applying any constraining force at the center of mass in the rest-frame of the gyroscope. This corresponds precisely to the Pirani supplementary condition. We therefore conclude that (37) gives $(d\mathbf{S}_0/dt)$ as measured by a co-moving observer when instrumental torques have been eliminated.

Even when an instrumental torque is present, however, a correction can be made for it. When the CP supplementary condition is used, the vector ϵ in the gyroscope rest-frame can be obtained by substituting the S_0^{i4} given by the last three of equations (28) into equation (14). This gives to first order

$$\epsilon_x = 0, \qquad \epsilon_y = vS_z/m_0, \qquad \epsilon_z = -vS_y/m_0.$$

Since $\mathbf{v}$ is here along x, this may be written in a rotation-covariant manner as

$$\epsilon = -(\mathbf{v} \times \mathbf{S})/m_0.$$

Thus when the CP condition is used, we must subtract the instrumental torque

$$-\epsilon \times \mathbf{F} = [(\mathbf{v} \times \mathbf{S}) \times \mathbf{F}]/m_0 = \mathbf{S}(\mathbf{v}\cdot\mathbf{f}) - \mathbf{v}(\mathbf{f}\cdot\mathbf{S})$$

from the rate of change of the gyroscope angular momentum given by the right side of (35). When this is done, equations (35) and (37) agree.

Effects of Earth Rotation and Extra-Terrestrial Objects.—Equation (37) gives the rate of change of angular momentum measured by a co-moving observer when the central attracting body (the earth) is spherically symmetrical and at rest. The effect of the axial rotation of the earth can be included by adding off-diagonal space-time components to the metric tensor. These components were computed by de Sitter[24] and by Lense and Thirring.[25] In the isotropic metric, with the vector angular velocity ω of the earth directed along the positive z axis, the additional components are $(x^1 = x, x^2 = y, x^3 = z)$:

$$g_{14} = -2I\omega y/r^3, \qquad g_{24} = 2I\omega x/r^3, \qquad g_{34} = 0, \tag{38}$$

where $I = 2mR^2/5$ is the moment of inertia of the earth of radius R, assumed to be homogeneous.

It is not difficult to see that the contribution of (38) to $(d\mathbf{S}/dt)$ is the same for the four spin equations of motion (18), (20), (23), and (24), and that it is not affected by the transformation to the gyroscope rest-frame. When this contribution is added to the right side, equation (37) may be written

$$(d\mathbf{S}_0/dt) = \mathbf{\Omega} \times \mathbf{S}_0, \tag{39}$$

where

$$\mathbf{\Omega} = {}^1/_2(\mathbf{f} \times \mathbf{v}) + (3m/2r^3)(\mathbf{r} \times \mathbf{v}) + (I/r^3)[(3\mathbf{r}/r^2)(\omega\cdot\mathbf{r}) - \omega], \tag{40}$$

Simple order of magnitude estimates based on equations (39) and (40) suffice to show that the effects of the moon, the sun, and the galaxy are negligible in comparison with the effect of the earth.

Interpretation of the Spin Equation of Motion.—The form of equation (39) shows that the magnitude of the spin angular momentum measured by a co-moving observer is constant. Thus, if the moment of inertia of the gyroscope does not change, the rotation frequency remains constant. As remarked above, the gyroscope then behaves like a clock that can be set to any desired frequency, and this frequency will exhibit Doppler and gravitational shifts when observed from outside.

It also follows from equation (39) that if a number of gyroscopes are traveling together with various magnitudes and directions for their angular momentum vectors, the angles between these vectors, as measured by a co-moving observer, remain constant. The spin axes of all the gyroscopes precess with the common vector angular velocity $\mathbf{\Omega}$ given by (40). This precession takes place with respect to the inertial frame, which is generally believed to be defined by the distant extragalactic nebulae, the so-called "fixed stars." Thus, an experimental verification of equations (39) and (40) will consist in essence of a series of comparisons of the direction of the spin axis of a gyroscope and particular "fixed stars," made at different times. If the gyroscope is in motion when a comparison is made, the well-known correction for aberration must be made.[26]

The first term on the right side of equation (40) is the Thomas precession,[27] which is a special relativity effect. It is independent of gravitation, and is present even when m is set equal to zero in (34) and (40). The second term is a consequence of Einstein's theory of gravitation that goes beyond the equivalence principle. Even though this term involves only the values of α and γ in (2), which can be obtained correctly from the equivalence principle, it also involves the equation of motion of matter of finite rest mass beyond the Newtonian approximation. It is interesting to note that if $\dot{\mathbf{v}}$ is infinitesimally small, so that $\mathbf{f}$ given by (34) is equal to $(m/r^3)\mathbf{r}$, the gravitational precession is three times the Thomas precession, and of the same sign. The effects produced by this second term in various special cases have been discussed by several authors, notably de Sitter,[28] Fokker,[29] and Pirani.[17] However, in none of these papers is the differential equation (39) or the full expression (40) for the time-varying precession angular velocity $\mathbf{\Omega}$ exhibited, nor is the relation between the different possible coordinate systems and supplementary conditions discussed.

The third term on the right side of (40) arises from the rotation of the earth, and is present even when the gyroscope is at rest. It has the interesting property that it is parallel to $\boldsymbol{\omega}$ at the poles ($\mathbf{r}$ parallel or antiparallel to $\boldsymbol{\omega}$), and antiparallel to $\boldsymbol{\omega}$ at the equator ($\mathbf{r}$ perpendicular to $\boldsymbol{\omega}$). This is physically plausible if we think of the moving earth as "dragging" the metric with it to some extent. At the poles, there is a tendency for the metric to rotate with the earth, and hence to cause the spin to precess in the direction of rotation of the earth. At the equator, we note particularly that the gravitational field, and hence also the dragging of the metric, falls off with increasing radial distance. If, then, we imagine the gyroscope oriented so that its axis is perpendicular to that of the earth, the side of the gyroscope nearest the earth is dragged with the earth more than the side away from the earth, so that the spin precesses in the opposite direction to the rotation of the earth. This third term is also a consequence of general relativity theory that goes beyond the equivalence principle, partly because the non-Newtonian equation of motion is required, and partly because the off-diagonal space-time components of the metric tensor cannot be inferred from the equivalence principle.

Experimental Consequences.—Two experimental arrangements especially commend themselves: a gyroscope in a satellite, and a gyroscope at rest in a laboratory fixed with respect to the earth.[30]

The satellite gyroscope is in free fall, so that $\mathbf{f} = 0$. Then for an orbit in the earth's equatorial plane, for example,

$$\boldsymbol{\Omega} = (3m/2r)\boldsymbol{\omega}_0 - (I/r^3)\boldsymbol{\omega}, \tag{41}$$

where $\boldsymbol{\omega}_0 = (\mathbf{r} \times \mathbf{v})/r^2$ is the instantaneous orbital angular velocity vector of the gyroscope. If m and r are to be expressed in cgs units, m/r must be replaced by Gm/c^2r, where G is the Newtonian gravitational constant and c is the speed of light. It is convenient to replace Gm by gR^2, where g is the acceleration of gravity at the surface of the earth. Then (41) may be written

$$\boldsymbol{\Omega} = (gR/c^2)[(3\boldsymbol{\omega}_0/2)(R/r) - (2\boldsymbol{\omega}/5)(R/r)^3], \tag{42}$$

where $gR/c^2 = 7.0 \times 10^{-10}$. It is easily seen that the second term of (42) is never more than a per cent or two of the first; thus $\boldsymbol{\Omega}$ is roughly parallel to $\boldsymbol{\omega}_0$ even when the orbit is not equatorial. For a satellite of moderate altitude, the precession is approximately 6×10^{-9} radians per revolution when the gyroscope spin axis is in the plane of the orbit.

The laboratory gyroscope is constrained to remain at rest with respect to the rotating earth, so that

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}, \qquad d\mathbf{v}/dt = \boldsymbol{\omega} \times \mathbf{v}. \tag{43}$$

The nongravitational constraining acceleration $\mathbf{f}$ may then be obtained from equations (34) and (43), and substituted into (40). The only experimental parameter in (40) is then the latitude λ of the laboratory. It is convenient to write $\boldsymbol{\Omega}$ in the form

$$\begin{aligned}\boldsymbol{\Omega} = (2gR\cos^2\lambda/c^2)[1 - (\omega^2R/4g)]\boldsymbol{\omega} + (2g\sin\lambda/c^2\omega)(\boldsymbol{\omega} \times \mathbf{v}) +\\ (2gR/5c^2)(3\sin^2\lambda - 1)\boldsymbol{\omega} - (6g\sin\lambda/5c^2\omega)(\boldsymbol{\omega} \times \mathbf{v}).\end{aligned} \tag{44}$$

The first line represents the first two terms of (40), and the second line the third term. Each line has been divided into a part proportional to $\boldsymbol{\omega}$, which results in a secular precesion, and a part proportional to $\boldsymbol{\omega} \times \mathbf{v}$, which contributes nothing to the time integral of $\boldsymbol{\Omega}$ over a whole number of days. Thus the secular precession arising from the effect of earth rotation on the metric vanishes when $\sin\lambda = 3^{-1/2}$, or $\lambda = 35°16'$. Since $\omega^2R/4g$, the ratio of centripetal to gravitational acceleration, is very small compared to unity, equation (44) shows that the secular part of $\boldsymbol{\Omega}$ is given to good approximation by

$$\boldsymbol{\Omega}_s = (4gR/5c^2)(1 + \cos^2\lambda)\boldsymbol{\omega}.$$

If the gyroscope spin axis is perpendicular to the earth's axis, the precession is approximately 3.5×10^{-9} $(1 + \cos^2\lambda)$ radians per day.

In comparing the two experimental arrangements, it should be noted that while the secular precessions per revolution are roughly the same, the period of a satellite of moderate altitude is much shorter than a day, so that the precession per unit time of the satellite gyroscope will be about 15 times that of the earth-bound gyroscope. Moreover, most of the experimental difficulties that seem to rise with a high-precision gyroscope, especially instrumental torques, are greatly reduced if the gyroscope does not have to be supported against gravity. On the other hand, it is much simpler at present to monitor a gyroscope in the laboratory than in a satellite.

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[6] Møller, C., *The Theory of Relativity* (Oxford: Oxford University Press, 1952), p. 326.

[7] The remarks in this paragraph have been made with particular force by Eddington, A. S., *The Mathematical Theory of Relativity* (Cambridge: Cambridge University Press, 1924), p. 105. It should be noted that β is no longer zero if the isotropic form of the metric is used.

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[20] Actually, Papapetrou (reference 15) derived the noncovariant equation (11) first, and then obtained the covariant form (9) from it.

[21] u_i will not precisely vanish in the particle rest-frame for a rotating central body, but can be neglected.

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