3 April 2000

ELSEVIER

Physics Letters A 268 (2000) 69–74

PHYSICS LETTERS A

www.elsevier.nl/locate/physleta

Global dynamics of cosmological expansion with a minimally coupled scalar field

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Received 20 July 1999; received in revised form 30 September 1999; accepted 23 February 2000
Communicated by A.R. Bishop

Abstract

We give a complete description of the asymptotic behavior of a Friedmann–Robertson–Walker Universe with 'normal' matter and a minimally coupled scalar field. We classify the conditions under which the Universe is or is not accelerating. In particular, we show that only two types of large time behavior exist: an exponential regime, and a subexponential expansion with the logarithmic derivative of the scale factor tending to zero. In the case of the subexponetial expansion the Universe accelerates when the scalar field energy density is dominant and the potential behaves in a specified manner, or if matter violates the strong energy condition $\rho + 3p > 0$. When the expansion is exponential the Universe accelerates, and the scalar field energy density is dominant. We find that for the Big Bang to occur at zero scale factor, the equation of state of matter needs to satisfy certain restrictions at large densities. Similarly, a never ending expansion of the Universe constrains the equation of state at small matter densities. © 2000 Elsevier Science B.V. All rights reserved.

PACS: 98.80.-k; 04.20.-q; 98.80.Hw; 95.30.Sf

In this Letter we give a complete generic description of the late time asymptotic behavior of the cosmological expansion under quite general conditions. While there is literature devoted to specific potentials and interesting attractor solutions within those models [1–3], in the present work we study the *generic* features of the late time asymptotics for *any* potential. We choose units in which $G = c = 1$.

Recent results from supernovae type Ia [4] indicate that the Universe may be accelerating. If this is the case, we are currently entering a period of cosmological inflation. An accelerating Universe is sometimes interpreted as evidence for a cosmological constant, which is not necessarily so. Rather, this testifies only that the dominant material in the Universe is characterized by an equation of state that satisfies $\rho + 3p < 0$ [5]. 'Exotic' matter described by this type of equation of state has been called quintessence [1]. A self-interacting scalar field is one way to provide acceleration without assuming this peculiar property of 'normal' matter [1–3]. As far back as ten years ago, Ratra and Peebles considered a cosmology where the scalar field energy density becomes dominant at the present cosmological epoch [2]. They found asymptotically stable equilibrium

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PII: S0375-9601(00)00151-1

solutions in which the scalar field energy density dominates the dynamics. Their work has recently been resurrected by Liddle and Scherrer [3]. These results correspond to certain special forms of the scalar field potential and also rest upon other assumptions.

Since the Universe is very homogeneous and isotropic on large scales, it can be approximated by a Friedmann–Robertson–Walker metric:

$$ds^2 = -dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{1}$$

where $d\Omega$ is the element of solid angle and $k=-1$, 0, or 1 according to whether the Universe is open, flat, or closed. In this work we consider mainly open and flat cosmologies. The only energy-momentum tensor consistent with homogeneity and isotropy is the one corresponding to a perfect fluid. Therefore we consider a Universe filled with material of energy density ρ and pressure p, and with a self-interacting scalar field φ that has potential $V(\varphi)$. We assume that

$$\rho > 0, \tag{2}$$

$$\rho + p > 0, \tag{3}$$

$$V(\varphi) \geq 0. \tag{4}$$

Those are fairly reasonable restrictions. Condition (2) states that the energy density is positive, while (4) ensures that the scalar field energy density never becomes negative. We believe it is unreasonable to consider material with pressure more negative than the vacuum ($p=-\rho$) and therefore we impose (3). We also exclude the vacuum case itself because the cosmological constant can be treated as part of the scalar field potential. The matter ρ in the Universe is made up of baryons, photons, neutrinos, dark matter, and whatever else it might be. We suppose the pressure to be some function of the density, $p = p(\rho)$. The equation of state in the present epoch is very close to that of a pressureless fluid ('dust'), but we consider an entirely general equation of state, assuming only (3) and

$$p(0) = 0 \tag{5}$$

(in the absence of matter there should be no pressure at all).

The field equations are

$$3\frac{\dot{R}^2}{R^2} = 8\pi\rho + \dot{\varphi}^2 + V(\varphi) - 3\frac{k}{R^2} \equiv \rho_T - 3\frac{k}{R^2}, \tag{6}$$

$$\ddot{\varphi} + 3\frac{\dot{R}}{R}\dot{\varphi} + \frac{1}{2}\frac{dV(\varphi)}{d\varphi} = 0, \tag{7}$$

$$\dot{\rho} = -\frac{3\dot{R}}{R}(\rho + p), \tag{8}$$

where the total energy density $\rho_T = 8\pi\rho + \rho_\varphi$, and $\rho_\varphi = \dot{\varphi}^2 + V(\varphi)$ is the energy density of the scalar field. The equations also combine to give

$$\begin{aligned} 3\frac{\ddot{R}}{R} &= -4\pi(\rho + 3p) + V(\varphi) - 2\dot{\varphi}^2 \\ &\equiv -4\pi(\rho + 3p) - \tfrac{1}{2}(\rho_\varphi + 3p_\varphi), \end{aligned} \tag{9}$$

where $p_\varphi = \dot{\varphi}^2 - V(\varphi)$ is the scalar field pressure. Thus the expansion is accelerating when $(\rho_\varphi + 3p_\varphi) < 0$ and larger in magnitude than $8\pi(\rho + 3p)$.

We consider an expanding (as opposed to contracting) Universe. Hence we choose the positive square root in Eq. (6), which, in view of Eqs. (2), (4) and $k = 0, -1$, implies $\dot{R}/R > 0$; hence $\dot{R} > 0$, so $R \to \infty$ as $t \to \infty$. We thus have a 4D dynamical system whose trajectories $\{R(t), \rho(t), \varphi(t), \dot{\varphi}(t)\}$ are specified by initial values $R_0 > 0$, $\rho_0 > 0$, φ_0, $\dot{\varphi}_0$ at some $t = t_0$.

The energy conservation Eq. (8) can be integrated to

$$\int_\rho^{\rho_0} \frac{d\xi}{\xi + p(\xi)} = 3\ln\left(\frac{R}{R_0}\right). \tag{10}$$

By Eqs. (8) and (3), $\rho(t)$ is a monotonically decreasing (positive) function, therefore it has a nonegative limit when $t \to \infty$. However, for large times $R \to \infty$, so the integral in the left-hand side of (10) must diverge. Because of our conditions (3) and (5), this can only happen if $\lim_{t\to\infty} \rho(t) = 0$; the matter density in the Universe goes monotonically to zero with its expansion. Note that the divergence of the integral (10) at $\rho = 0$ requires the pressure go to zero fast enough for small densities (c.f. the integral converges if $p \propto \rho^\gamma$, $\gamma < 1$ as $\rho \to +0$). It is sufficient for the divergence that the derivative $p'(0)$ exists, which we assume in the sequel.

We now briefly turn to the Big Bang (BB). We require that $R_0 \to 0$ at the time (either finite or $-\infty$) of the BB singularity in the past. As $R_0 \to 0$, $\ln(R/R_0) \to \infty$, and again the integral in (10) must diverge, which is impossible unless $\rho_o \to \infty$ (a definition of the BB singularity). However, this integral does not diverge at $\rho_0 = \infty$ for any thinkable dependence $p(\rho)$: for the divergence, the pressure must not grow too fast at large densities. Thus a surprising fact is that the very existence of the BB imposes a limitation on the equation of state in the early Universe. In terms of a power scale, if $p \propto \rho^\gamma$ for $\rho \gg 1$, then $\gamma \le 1$. It is suggestive that if $\gamma > 1$ not only does the integral converge, but causality is violated: the speed of sound is

$$c_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_S \propto \gamma \rho^{\gamma - 1},$$

so it becomes greater than one for sufficiently large ρ.

Once again, for the BB to occur at $R_0 = 0$ and/or the infinite expansion to exist,the equation of state must ensure that the integral (10) diverges at $\rho_0 = \infty$ and/or $\rho = 0$. This result does not depend on the scalar field system, as it is minimally coupled; in particular, it is valid for $V \equiv 0$ as well.

While certainly thought provoking, these considerations about the BB should be taken with caution as the situation in the very early universe is probably considerably more complicated. In fact, at very early times the universe probably underwent a period of inflation [5–7]. In such a period the evolution of the universe is not dominated by the energy density of regular matter, but by the energy density of the inflation field(s). In fact inflationary scenarios in which the expansion is exponential violate the weak energy condition (3). Since inflation is the only known solution to the horizon problem [5–7], our assumptions are probably too simplistic when applied to an early universe which underwent an exponential inflationary phase. On the other hand, accelerated sub-exponential expansion that solves the horizon problem can occur in the early universe and condition (3) can be satisfied. Global constraints regarding the weak energy condition and generic inflationary scenarios have recently been studied by Vachaspati and Trodden [8]. Whether the existence of the BB at $R_0 = 0$ implies an equation of state which is causal is a matter of a separate investigation.

In order to study the limit of large times for the cosmological dynamical system (6)–(8), we first note that Eqs. (8) and (7) combine into

$$\rho_{\dot{T}} = -\frac{6\dot{R}}{R}\left[4\pi(p+\rho) + \dot{\varphi}^2\right] < 0, \tag{11}$$

with the inequality implied by (3) (it is straightforward to see that the density does not turn to zero at any finite time). This shows that the initial store of energy is dissipated into the expansion, and that, just as ρ, the total energy density ρ_T is a (positive) decreasing function of time; it is a Lyapunov function of our dynamical system (e.g. Ref. [9], 2.3). Thus ρ_T has a nonegative limit $\lim_{t\to\infty} \rho_T = \rho_T^\infty \equiv \rho_T^\infty(R_0, \rho_0, \varphi_0, \dot{\varphi}_0) \ge 0$; since $\rho \to 0$, we have also $\lim_{t\to\infty} \rho_\varphi = \rho_T^\infty$. Therefore the large time behavior of the cosmological expansion is described by

$$3\frac{\dot{R}^2}{R^2} = \rho_T^\infty + \dots, \tag{12}$$

$$\rho \to 0, \tag{13}$$

$$\rho_\varphi = \dot{\varphi}^2 + V(\varphi) = \rho_T^\infty + \dots, \tag{14}$$

where dots stand for the terms tending to zero; in addition, Eqs. (7) and (9) become

$$\ddot{\varphi} + \beta\dot{\varphi} + \frac{1}{2}\frac{dV(\varphi)}{d\varphi} + \dots = 0, \quad \beta \equiv \sqrt{3\rho_T^\infty} \ge 0, \tag{15}$$

$$3\frac{\ddot{R}}{R} = \rho_T^\infty - 3\dot{\varphi}^2 + \dots = 3V(\varphi) - 2\rho_T^\infty + \dots. \tag{16}$$

Depending on whether ρ_T^∞ is positive, there are clearly two different large time regimes of the expansion.

If $\rho_T^\infty > 0$, the Universe expands (and of course accelerates) exponentially; this is the only possible regime when the potential is bounded away from zero, $V(\varphi) \ge V_{\min} > 0$, for instance, if a cosmological constant exists.

The expansion is subexponential with $\dot{R}/R \to 0$ when $\rho_T^\infty = 0$, which requires the zero value to belong to the closure of the $V(\varphi)$ image. In this case, the Universe could accelerate or decelerate depending on further properties of the scalar field potential.

We first consider the subexponential case, i.e., any solution with $\rho_T^\infty = 0$.

Obviously, here $\lim_{t\to\infty}\dot{\varphi}(t) = 0$, $\lim_{t\to\infty}V(\varphi(t)) = 0$, so that the scalar field also goes to some limit φ_∞, either finite or infinite, and $V(\varphi_\infty) = 0$. If the limit is finite, $|\varphi_\infty| < +\infty$, then, just by inequality (4), $dV(\varphi)/d\varphi|_{\varphi=\varphi_\infty} = 0$, $d^2V(\varphi)/d\varphi^2|_{\varphi=\varphi_\infty} > 0$; the system evolves towards one of its finite stable critical (fixed) points with the zero minimum of the potential. It is not difficult to show that the expansion in this case is not accelerating unless $p'(0) < -1/3$, which means $\rho + 3p < 0$ at least for small enough densities; the scalar field thus cannot 'outweigh' the 'normal' matter under such conditions.

However, if $\varphi_\infty = \pm\infty$, which requires $\lim_{\varphi\to\pm\infty}V(\varphi) = 0$, the situation with the acceleration can be different depending on the details of behavior of the potential at infinity (the system in any case goes again to a fixed point, only corresponding to an infinite value of the field). We describe here two classes of potentials allowing the acceleration.

(i) Potentials vanishing exponentially at infinity. For some $a > 1$, let

$$V(\varphi) = a(3a-1)\exp(-2\varphi/\sqrt{a})[1 + o(1)]$$

as $\varphi \to +\infty$, and this asymptotic formula may be differentiated. If at large times φ goes to infinity, then for $t \to \infty$,

$$\varphi \simeq \sqrt{a}\ln(t), \quad \rho + 3p \sim (1+3\nu)/t^{3a(1+\nu)}$$

with $\nu = p'(0)$, and, by (9), $\ddot{R}/R \simeq a(a-1)/t^2$ if $3a(1+\nu) > 2$, which is true for $1 + 3\nu > 0$, i.e., exactly when the strong energy condition $\rho + 3p > 0$ holds for small densities. The expansion in this case accelerates.

(ii) Potentials vanishing as a power at infinity. For some $b > 0$, let

$$V(\varphi) = [4/(b+4)]^2(\sqrt{b}/\varphi)^b[1 + o(1)]$$

as $\varphi \to +\infty$, and this asymptotic formula may be differentiated. If at large times φ goes to infinity, then for $t \to \infty$,

$$\varphi \simeq \sqrt{b}\, t^{2/(b+4)},$$

$$\rho + 3p \sim (1+3\nu)\exp[-3(1+\nu)t^{4/(b+4)}]$$

with $\nu = p'(0)$, and, by (9), $\ddot{R}/R \simeq [4/(b+4)]^2 t^{-2/(b+4)}$, so that the expansion accelerates.

These two types of potentials were the ones studied in [1–3].

To make the picture complete, we consider now the (not that interesting) exponential regime of expansion, i.e., solutions with $\rho_T^\infty > 0$.

The large time behavior of the scalar field is described in this case by the damped anharmonic oscillator Eq. (15). According to the Poincaré–Bendixson theory of autonomous dynamical systems on the plane (e.g. [9], 2.8), every bounded non-closed phase trajectory has a limit trajectory which is either a critical point or an isolated closed orbit (limit cycle, periodic solution). However, Eq. (15) has no periodic solutions at all. Indeed, it can be rewritten as

$$\rho_{\dot{\varphi}} = -2\beta\dot{\varphi}^2,$$

and the assumption that $\varphi(t)$ is periodic with some period $T > 0$ implies

$$\int_0^T \rho_{\dot{\varphi}}\, dt = 0 = -2\beta\int_0^T \dot{\varphi}^2\, dt,$$

that is, $\dot{\varphi}(t) \equiv 0$, because $\beta > 0$. The derivative of the scalar field is always bounded, $\dot{\varphi}^2(t) < \rho_T(t_0)$ for $t > t_0$. Therefore if the scalar field is bounded, then the solution goes to a critical point, $\lim_{t\to\infty}\dot{\varphi}(t) = 0$, $\lim_{t\to\infty}\varphi(t) = \varphi_\infty$, $|\varphi_\infty| < +\infty$, $V(\varphi_\infty) = \rho_T^\infty > 0$, and, by (15), $dV(\varphi)/d\varphi|_{\varphi=\varphi_\infty} = 0$, $d^2V(\varphi)/d\varphi^2|_{\varphi=\varphi_\infty} > 0$. Again the field, and with it the whole system, evolves towards a finite stable critical point which now corresponds to a non-zero minimum of the potential. This is the only possibility for a system with a growing potential, $\lim_{\varphi\to\pm\infty}V(\varphi) = +\infty$, because under this condition all trajectories are obviously bounded. The system in such a case is dissipative, and its limit behavior is conveniently described in terms of the connected global attractor consisting of all the critical points and (heteroclinic) trajectories connecting the unstable critical points with the stable ones [10,11].

It remains thus to find out about the unbounded phase trajectories of (15), if they exist at all, that is, under the condition that $V(\varphi)$ is bounded at least at one of the infinities. As explained, the unbounded trajectory means that $\lim_{t\to\infty}\varphi(t) = \pm\infty$; to be precise, let us speak about the positive infinity. If the limit of the potential, $\lim_{\varphi\to+\infty}V(\varphi) = V_\infty < +\infty$ exists, then clearly $\lim_{t\to\infty}\dot\varphi(t) = 0$, $V_\infty = \rho_T^\infty$, so that the whole system once again tends to a stable fixed point, with just an infinite limit value of the scalar field.

The exceptions of evolution towards a critical point thus could possibly be provided only by the potentials which are bounded but have no limit at infinity (it looks rather peculiar from the physical point of view, unless they are periodic). Our survey of the results pertinent to this case so far has shown that one cannot completely rule out the solutions which undergo rather strange damped oscillations characterized by the following: $\lim_{t\to\infty}\varphi(t) = \pm\infty$, $\dot\varphi(t)$ is bounded and has no limit (in particular, $\lim_{t\to\infty}\dot\varphi(t) \neq 0$), $\lim_{t\to\infty}\rho_\varphi(t) = \rho_T^\infty > 0$, and

$$\int_\tau^\infty \dot\varphi^2\, dt < +\infty, \quad \tau \geq t_0 .$$

If these exist at all, the relevance of such regimes to the Universe appears to be questionable at the very least.

Let us now give a list of the most significant facts regarding the large time behavior of the cosmological expansion we have so far established.

1. The matter density decreases to zero, the total energy density decreases to a (nonnegative) constant, the scalar field energy density is non-increasing and goes to the same constant as the total density, always.

2. Except possibly for such potentials $V(\varphi)$ that are bounded at infinity but have no limit there, the dynamical system always evolves towards one of its critical (fixed) points with either finite or infinite limit value φ_∞ of the scalar field; the time derivative of the field goes to zero in both cases; if the limit value of the field is finite, the potential has a minimum at it.

3. The expansion regime at large times is either exponential or subexponential with $\dot R/R \to 0$; in the (typical) case of evolution to a fixed point the former occurs if $V(\varphi_\infty) > 0$, the latter if $V(\varphi_\infty) = 0$.

4. The subexponential expansion may be accelerating, that is, the scalar field can dominate 'normal' matter ($\rho + 3p > 0$), provided that the scalar field tends to infinity and the potential has a zero limit $V(\varphi_\infty) = V(\infty) = 0$ and certain asymptotic behavior there.

Properties 1–3 of the cosmological expansions are hardly surprising, however, all these results are now firmly and unambiguously established under general conditions of a clear physical origin.

Given the potential, the choice between the described possibilities is made by the initial conditions, so for the same potential the final stage of the expansion might be different depending on what happened in the early Universe. Let us illustrate this by some examples.

If $V(\varphi)$ is a potential well, i.e., it has a single minimum at some $\varphi = \varphi_*$ and goes (however slowly!) to infinity on both sides of it, all the solutions, disregarding the initial conditions they stem from, tend to the single (stable) equilibrium at φ_*; depending on whether $V(\varphi_*) = 0$ or not, it is either a subexponential regime or an exponential one.

A much more sophisticated picture arises with, say, the potential plotted in Figure 1. It has a zero minimum at $\varphi = \varphi_2$, a non-zero minimum at $\varphi = \varphi_4$, two unstable equilibria (maxima) at $\varphi = \varphi_{1,3}$, $\varphi_1 < \varphi_2 < \varphi_3 < \varphi_4$, goes to infinity as $\varphi \to +\infty$ but tends to zero when $\varphi \to -\infty$. Both exponential ($\varphi_\infty = \varphi_4$) and subexponential ($\varphi_\infty = \varphi_2, -\infty$) regimes are possible, depending on how the expansion starts; the latter regime with $\varphi_\infty = -\infty$ can be accompanied by acceleration even if $\rho + 3p > 0$, provided that $V(\varphi)$ has the right asymptotic behavior at $\varphi \to -\infty$. The exponential growth occurs, for instance, when $\varphi_0 > \varphi_3$ and $\rho_\varphi(t_0) < V(\varphi_3)$; subexponential expansion

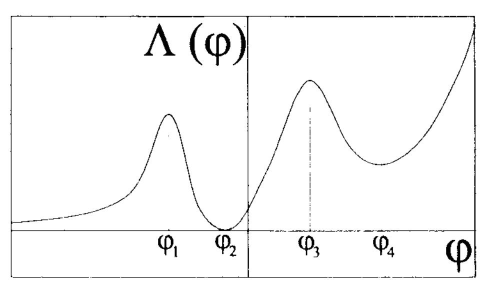

Fig. 1.

without acceleration happens for the initial conditions with $\varphi_1 < \varphi_0 < \varphi_3$ and $\rho_\varphi(t_0) < V(\varphi_1) < V(\varphi_3)$ ($\varphi \to \varphi_2$); subexponential regime with possible acceleration takes place when $\varphi_0 < \varphi_1$ and $\rho_\varphi(t_0) < V(\varphi_1)$, when $\varphi \to -\infty$.

Our final remark is about the closed Universe, $k = 1$. The only thing we need to validate our results in this case is that the expansion never stops (and maybe turns into contraction), i.e., that $\dot{R} > 0$ for any $t > t_0$. By (6), we are speaking about solutions for which $\rho_T - 3/R^2$ is positive for all $t > t_0$ as soon as it is positive at the initial moment $t = t_0$; for any such 'infinite expansion solution', our results are true. There are many conditions under which these solutions exist. For instance, if $V(\varphi) \geq V_{min} > 0$, then, since $\rho_T > V(\varphi)$ by definition, any solution starting with $R_0 > R_{min} = \sqrt{3/V_{min}}$ is an infinite expansion solution. The other condition which does not depend on the potential at all is derived from the fact that, by the same definition, $\rho_T > 8\pi\rho$. The energy conservation Eq. (10) shows that at large times $\rho \sim R^{-3(1+\nu)}$, $\nu \equiv p'(0)$. If $p'(0) < -1/3$, then $-3(1+\nu) < 2$, and any solution with large enough 'starting radius' R_0 satisfies $\rho_T > 3/R^2$ and hence is an infinite expansion solution, etc.

Therefore it seems that acceleration of the Universe is a fairly general behavior. What is surprising in light of recent supernovae results [4] is that the period of acceleration is just starting around the present time. One would have to fine-tune the parameters of our potentials to account for this. This rather unfortunate situation would be resolved if we had an appropriate potential which came from more fundamental physics.

A paper with similar results for the technically much more complicated case of scalar-tensor theories of gravity with non-minimal coupling of the scalar field, is now in preparation.

Acknowledgements

This work was supported by NASA grant NAS 8-39225 to Gravity Probe B. We are grateful to R.V. Wagoner for many valuable comments and to the Gravity Probe B Theory Group for fruitful discussions. We would also like to thank Manuel E. Cid and Ricardo Díaz Díaz for their help with the figure.

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