SUPPORT DEPENDENT TORQUES IN THE RELATIVITY GYROSCOPE EXPERIMENT

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1. INTRODUCTION

As explained in the preceeding paper by J.P. Turneaure et.al. the Relativity Gyroscope Experiment is designed to measure the precession rate of an electrostatically supported mechanical gyroscope to an accuracy of better than one milliarc-second/year. The General Theory of Relativity predicts that for a gyroscope in a 650 km polar orbit the geodetic effect, due to the interaction of the gyroscope with its orbital motion about the earth, will produce a drift rate of 6.6 arc-seconds/year; while the motional or gravitomagnetic effect, due to the interaction of the gyroscope with the spinning earth, will produce a drift rate of 44 milliarc-seconds/year. To achieve this accuracy, it is necessary to ensure that drift rates of the gyroscope due to classical effects are less than one milliarc-second/year or can be modelled and subtracted from the data to this accuracy.

The most important class of the classical torques are the support dependent torques due to the electrostatic suspension system. Although the gyroscopes will be flown in a drag-free satellite where the residual accelerations of the proof mass are less than 10^{-10} g, an analysis of the classical torques on the gyroscope[1] suggests that these torques due to the electrostatic suspension system will be dominant. Therefore, a careful analysis of the support dependent torques on the gyroscope must be used to place requirements on the gyroscopes and the spacecraft such that the support dependent torques produce a drift rate less than 0.3 milliarc-seconds/year. This analysis is also useful for ground-based testing and evaluation of gyroscopes: a comparison of the predicted and observed drift rates will not only confirm the results of this analysis but will be an additional means of qualifying gyroscopes for flight.

Earlier work on the suspension torques of electrostatically supported gyroscopes was done by Matchett[2] and later extended by P. Eby[3]. They calculated the electrostatic force on a small surface element of the spherical rotor and then found the torque on this surface element by taking the cross product of this force with a vector from the center of mass of the rotor. The net torque on the rotor was then found by transforming to a laboratory-fixed reference frame and integrating the torque over the electrode surface. To simplify the analysis Matchett and Eby assumed that the shape of the rotor could be averaged over its rotation and then expanded this average shape in a cosine series. The torque on the rotor could then be calculated for each of the harmonics in this cosine series. Eby used numerical integration to evaluate the torques on the rotor up to the twentieth harmonic for the electrode shape planned for the Relativity Gyroscope Experiment.

Recent work at Stanford has shown that another approach to calculating the electrostatic torques on a gyroscope confirms the results of this earlier work and gives additional insight into the nature of these electrostatic torques. Instead of calculating the torque due to each small surface element of the rotor and integrating over the electrode to find the net torque, the torque may be found by calculating the energy stored in the electrostatic field and differentiating this expression with respect to one of the rotation angles to find the torque on the rotor. The energy stored in the electrostatic field is found in terms of the capacitance between the electrodes and the rotor. For an arbitrary rotor and electrode shape, the capacitance may be found by expanding the radius of the rotor and the shape of the electrode in spherical harmonics and then calculating the capacitance in terms of the amplitudes of the spherical harmonics. In performing this calculation, it has been extremely helpful to borrow some of the mathematical methods which have been developed for quantum mechanics[4]: Rotation operators, or D-matrices, are used to transform from one reference frame to another and the angular momentum operator is used to differentiate these expressions with respect to the rotation angles. The results of the calculations are analytical expressions for the suspension torques for an arbitrary rotor and electrode shape and for an arbitrary orientation of the rotor with respect to the housing.

The second part of this paper is a calculation of the electrostatic torque on a rotor where the electrodes and the housing are fixed with respect to a laboratory reference frame. The third part discusses the extension of the calculations to the specific experimental configuration planned for the Relativity Gyroscope Experiment. The final section discusses application of these calculations to ground-based testing and evaluation of gyroscopes.

2. ELECTROSTATIC TORQUES FOR A FIXED HOUSING

The total energy stored between two plates of a capacitor is

$$U = 1/2\ CV^2 \tag{1}$$

where C is the capacitance, and V is the potential difference between the two capacitor plates. As long as the radius of curvature of the rotor or the electrode is much larger than the gap between the electrode and the rotor, the capacitance is given by[5]

$$C = \int_{\Omega_i} \frac{\varepsilon_0 d\Omega}{d} \qquad (2)$$

where the integral is over the solid angle, Ω_i, beneath the electrode, ε_0 is the permittivity of free space, and d is the gap between the electrode and the rotor, which will be a function of the location specified by the angular coordinates, θ and ϕ. The gap, d, may be written as the sum of a nominal gap, d_0, which is independent of the location and a small variation Δd, which depends on the location

$$d_0 = d_0 + \Delta d \qquad (3)$$

Assuming that Δd is much less than d_0, the capacitance may be written as a series

$$C = C_o + C_1 \qquad (4)$$

where

$$C_0 = \frac{\varepsilon_0}{d_0} \int_{\Omega_i} d\Omega$$

$$(5)$$

$$C_1 = -\frac{\varepsilon_0}{d_0} \int_{\Omega_i} \frac{\Delta d(\theta, \phi)}{d_0} d\Omega$$

Each of the terms in this series is smaller than the previous term by an amount $\Delta d/d_0$.

To evaluate the integrals it is necessary to specify the electrode shape and the shape of the rotor. Since each integral is over only that portion of the surface beneath the electrode, it is convenient to define an electrode shape function in a coordinate system fixed with respect to the electrode as

$$f_i(\theta, \phi) = \begin{cases} 1 & \text{beneath the electrode} \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

This electrode shape function may then be expanded in terms of the spherical harmonics, $Y_{\ell m}(\theta, \phi)$

$$f_i(\theta, \phi) = \sum_{j=0}^{\infty} \sum_{m=-\ell}^{+\ell} f_{\ell m} Y_{\ell m}(\theta, \phi) \qquad (7)$$

The electrode shape is completely specified by coefficients, $f_{\ell m}$. Evaluation of the integrals in Equation (5) is then considerably simplified if the integrand is multiplied by the electrode shape function, and the integral is evaluated over the entire solid angle. Then, the orthonormal properties of the spherical harmonics may then be used to evaluate these integrals.

The variation in the gap between the rotor and the electrode is the difference between the radius of the electrode and the radius of the rotor:

$$\Delta d(\theta, \phi) = \Delta R(\theta, \phi) - \Delta r(\theta, \phi) \tag{8}$$

Here $\Delta R(\theta, \phi)$ is the difference between the radius of the electrode at the location (θ, ϕ) and its nominal radius and $\Delta r(\theta, \phi)$ is the difference between the radius of the rotor and its nominal radius. The variation in the electrode radius may be expanded in terms of spherical harmonics

$$\Delta R(\theta, \phi) = \sum_{k=1}^{\infty} \sum_{m=-k}^{+k} R_{km} Y_{km}(\theta, \phi) \tag{9}$$

in a coordinate system which is fixed with respect to the electrodes.

The variation in the rotor radius may also be expanded in terms of spherical harmonics in a coordinate system which is fixed with respect to the rotor.

$$\Delta r(\theta', \phi') = \sum_{\ell=1}^{\infty} \sum_{p=-\ell}^{+\ell} r_{\ell p} Y_{\ell p}(\theta', \phi') \tag{10}$$

Here θ' and ϕ' are the spherical coordinates of a point on the surface of the rotor with $\theta'=0$ defined as the direction of the instantaneous spin axis. To express the rotor radius in coordinate system fixed with respect to the housing, the rotation matrices may be used to transform the spherical harmonics from the rotor-fixed reference frame to the electrode-fixed reference frame:

$$Y_{\ell p}(\theta', \phi') = \sum_{q=-\ell}^{+\ell} D^{\ell}_{qp}(\alpha, \beta, \gamma)\, Y_{\ell q}(\theta, \phi) \tag{11}$$

The properties of these rotation matrices are discussed in more detail in Reference 4. Here α, β and γ are the Euler angles which define the orientation of the rotor-fixed reference frame with respect to the electrode-fixed reference frame: α and β are the spherical coordinates of the rotor spin axis, and the angle γ represents a rotation about that spin axis. Combining two equations (10), and (11) the variation in the rotor radius in the electrode-fixed reference frame is:

$$\Delta r(\theta, \phi) = \sum_{\ell=1}^{\infty} \sum_{p=-\ell}^{+\ell} \sum_{q=-\ell}^{+\ell} r_{\ell p} D^{\ell}_{qp}(\alpha, \beta, \gamma)\, Y_{\ell q}(\theta, \phi) \tag{12}$$

The above expressions for the electrode shape function, the variation in the electrode radius, and the variation in the rotor radius may be used to evaluate the integrals in Equation (5), using the ortho-normal properties of the spherical harmonics.

$$C_0 = \frac{\varepsilon_0 r_0^2}{d_0} \sqrt{\frac{1}{4\pi}}\, f_{oo} \tag{13}$$

$$C_1 = -\frac{\varepsilon_0 r_0^2}{d_0^2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{+\ell} f_{\ell m}(R_{\ell m} - \sum_{p=-\ell}^{+\ell} r_{\ell p}\, D^{\ell}_{mp}(\alpha,\beta,\gamma))$$

Higher order terms in this series may also be evaluated.

The torque on a gyroscope is evaluated by differentiating the expression for the energy stored in the electrostatic field with respect to one of the three Euler angles. The result is the component of the torque along the axis of rotation of the Euler angle.[6] Letting η represent one of the three components of the Euler angles, the component of the torque along the axis of rotation of the angle η is

$$\tau_\eta = \frac{1}{2} V^2 \frac{\partial C}{\partial \eta} \tag{14}$$

Since the first term in the series expansion of the capacitance is independent of the Euler angles, it makes no contribution to the torque on the rotor. The next term in the series expansion for the capacitance does depend on the Euler angles. Torques which arise from this term are called the primary torques; torques due to the next term in the series expansion for the capacitance are called secondary torques.[2]

With the resulting expressions for the torque on the rotor it is straightforward to demonstrate a number of interesting points. The electrodes which will be used in the relativity gyroscope experiment are circular electrodes deposited on the inner spherical surface of the housing. Taking the z-axis to be the symmetry axis of the electrodes, the only nonzero coefficients in the expansion of the electrode shape function, $f_{\ell m}$ are those coefficients for which m=0. In addition, the coefficients for the electrode along the negative z-axis, $f_{\ell o}^-$ are related to those coefficients for the electrode along the positive z-axis $f_{\ell o}^+$ by the relation

$$f_{\ell o}^- = (-1)^{\ell} f_{\ell o}^+ \tag{15}$$

If there is no variation in the voltage applied to the electrodes at the rotor spin speed, then it is possible to average over the rotation of the gyroscope. In this case, all the coefficients for the spherical harmonic expansion of the rotor shape, $r_{\ell p}$ with $p \neq 0$ average to zero.

With these assumptions the net primary torque due to the two electrodes along the z-axis is

$$\tau_\beta = \frac{\varepsilon_0 r_0^2}{2d_0^2} \sum_{\ell=1}^{\infty} (V_+^2 + (-1)^\ell V_-^2)\, f_{\ell o}^+\, r_{\ell o} \frac{\partial}{\partial\beta} P_\ell(\cos\beta) \tag{16}$$

where V_+ is the potential difference between the electrode along the positive z-axis and the rotor and V_- is the potential difference between the electrode along the negative z-axis and the rotor. The identity $D^\ell_{oo}(\alpha,\beta,\gamma) = P_\ell(\cos\beta)$ has been used to arrive at the above result. The components of the primary torque along the α and γ rotation axes are identically zero.

Thus, the primary torques depend only on the asphericities of the rotor averaged over the rotation axis, $r_{\ell o}$. The secondary torques also depend on the asphericities of the rotor but are also proportional to the ratio $R_{\ell m}/d_o$. Hence, if the asphericity of the housing or the miscentering of the rotor is small compared to the nominal gap, d_o, the requirements on the sphericity of the rotor must generally be stricter than those on the sphericity of the housing or the miscentering of the rotor. The above expression for the torque shows that for circular electrodes the primary torques tend to make the rotor spin axis precess about the direction of the electrode symmetry axis. Also, the torque due to the odd harmonics of the rotor shape depend on the difference between the squares of the voltages applied to the electrodes, while the torques due to the even harmonics of the rotor shape depend on the sum of the squares of the applied voltages. If the same voltage is applied to both electrodes, then the torques due to the odd harmonics of the rotor shape are zero.

3. RELATIVITY GYROSCOPE EXPERIMENT

Using these techniques, the support dependent torques have been calculated for the planned experimental configuration for the Relativity Gyroscope Experiment[7]. In this case, the housing is no longer fixed but is allowed to roll about the line of sight to the reference star with a 10 minute period, and the average torque is calculated over the roll period. The four gyroscopes are initially spun up to 170 Hz along the housing roll axis to within to 5 arc seconds. Subsequent misalignments between the housing roll axis and the rotor spin axis will be due to the aberration of starlight due to the orbital motion of the satellite about the earth (amplitude 5 arc sec), the aberration of starlight due to the orbital motion of the earth about the Sun (amplitude 20 arc sec), and the relativistic effects. For each gyroscope the housing roll axis will lie at the midpoint between two of the electrode axes and perpendicular to the third electrode axis.

The dominant support dependant torques for this planned experimental configuration may be placed in six separate categories. The primary torques are calculated (1) for the case where the rotor spin axis is exactly aligned with the housing roll axis and the housing roll axis is exactly aligned with the midpoint between the two electrode axes, (2) for the case where the rotor spin axis is slightly misaligned with the housing roll axis, and (3) for the case where the housing roll axis does not quite lie at the midpoint between two of the electrode axes. The secondary torques have been calculated for the case where the housing roll axis and the rotor spin axis are exactly aligned as in case (1) above but there is (4) a constant miscentering of the rotor with respect to the housing (5), a miscentering of the rotor which varies at the housing roll frequency, or (6) an asphericity of the housing. The additional secondary torque for the case where the rotor spin axis or the housing roll axis is slightly misaligned is expected to be at least a factor of 100 smaller than these torques listed above. Higher order contributions to the torques are also expected to be significantly smaller than the torques given above.

With these results the torques may be calculated if the voltages applied to each of the six electrodes in known. This voltage is determined by the average (or preload) voltage applied to two electrodes along a given axis and the residual acceleration of the rotor. Sources of acceleration on the rotor which have to be considered are (1) the acceleration due to the gradient in the earth's gravitational field since the rotor is not at the same location as the drag-free proof mass, (2) acceleration due to noise or residual bias in the drag-free controller, (3) acceleration due to noise in the rotor's electrostatic suspension system, (4) acceleration due to noise in the satellite's attitude control system, (5) acceleration due to the self-gravitation field of the satellite, and (6) centrifugal acceleration caused if the center of the gyroscope does not lie exactly on the satellite roll axis.

The calculations yield analytical expressions for the dominant support dependent torques which were discussed above. For example, consider the primary torques for the case where the spin axis is aligned with the housing roll axis, and the housing roll axis lies at the midpoint between two of the electrode axes. The torques due to the odd harmonics of the rotor shape are[7]

$$\tau = m q_p(\Omega)\, a_{odd} \qquad (17)$$

where m is the rotor mass, $q_p(\Omega)$ is the component of the acceleration perpendicular to the housing roll axis at the roll frequency, Ω, as seen in a housing fixed reference frame, and a_{odd} is an easily calculated dimension which depends on the odd harmonic contributions to the asphericity of the rotor. The torques due to the even harmonics of the rotor shape, for the case where the housing roll axis lies midway between the a and b electrode axes is

$$\tau = \frac{m}{2}\left[h_a(\Omega) - h_b(\Omega) + \frac{q_a^2}{h_a}(\Omega) - \frac{q_b^2}{h_b}(\Omega)\right] a_{even} \qquad (18)$$

Here, $h_a(\Omega)$ and $h_b(\Omega)$ are the roll frequency components of the preload acceleration along the a and b electrode axes and $q_a^2(\Omega)$ and $q_b^2(\Omega)$ are the roll frequency components of the square of the physical acceleration along these two axes. The preload acceleration is proportional to the square of the average voltage applied to the two electrodes along a given axis. It is equal in magnitude to the physical acceleration when the control system causes the voltage on one of the electrodes to be zero.

These two expressions for the torques on the gyroscope lead to a set of requirements on the rotor and the satellite such that no one of these support dependant torques is strong enough to cause a drift rate greater than 0.3 milliarc seconds/year. For example the asphericity of a flight quality rotor is expected to be such that

$$a_{odd} < 10^{-6} \text{ cm}$$
$$a_{even} < 10^{-5} \text{ cm}$$

The larger contribution to the asphericity from the even harmonics of the rotor shape is due to the centrifugal distortion of the rotor. With these restrictions on the asphericity of the rotor, the requirements on the acceleration of the rotor are:

$$q_p(\Omega) < 10^{-10} \text{ g}$$
$$1/2\ [h_a(\Omega) - h_o(\Omega)] < 10^{-11} \text{ g}$$
$$\frac{1}{2}\left[\frac{q_a^2(\Omega)}{h_a} - \frac{q_b^2(\Omega)}{h_b}\right] < 10^{-11} \text{ g}$$

Obviously, there will be a trade-off between these requirements: tightening the requirements on the rotor will loosen the requirements on the acceleration and vice versa.

Similar expressions may be derived for the torques due to the even and odd harmonics of the rotor shape for the primary torques where the spin axis or the roll axis is misaligned and for the secondary torques. Requiring that no one of these torques produce a drift rate greater than 0.3 milliarc-seconds/year leads to a set of constraints on the gyroscope and the spacecraft. Since these expressions for the torque are analytical it is clear from these results what the trade-offs between the various requirements will be. For example, tightening the requirements on the sphericity of the rotor will loosen the requirements on the drag-free control system. An example of the requirements on the gyroscopes and the spacecraft which is sufficient to meet the above criteria is given in Table 1.

TABLE 1

EXAMPLE OF THE REQUIREMENTS ON THE GYROSCOPES AND SPACECRAFT

This table is an example of a set of requirements on the gyroscopes and the drag-free control system which satisfy the condition that none of the calculated electrostatic torques on the gyroscope produce a drift rate greater than 0.3 milli-arc-seconds/year. In this example, the specifications of one gyroscope are assumed to have certain values, and the requirements on the spacecraft drag-free control system are derived from the analytical expressions for the torque. The requirement on the accelerations are given in a reference frame fixed in the spacecraft.

<u>Gyroscope</u>

1.	Rotor spin-axis misalignment	< 20 arcsec.
2.	Housing roll axis misalignment	< 3 arc min.
3.	Miscentering of the rotor	< 10 microin.
4.	Variation in miscentering at roll frequency	< 0.1 microin.
5.	Amplitude of housing asphericity	< 10 microin.
6.	Amplitude of rotor asphericity	< 0.4 microin.
7.	Electrode-to-rotor gap	1 milli-in.
8.	Rotor spin speed	170 Hz
9.	Preload acceleration	2×10^{-7} g

<u>Drag-Free Control System</u>

1.	D.C. acceleration parallel to the roll axis	$< 10^{-7}$ g
2.	Acceleration perpendicular to the roll axis at the roll frequency	$< 10^{-10}$ g
3.	Acceleration parallel to the roll axis at the roll frequency	$< 10^{-9}$ g
4.	Acceleration parallel to the roll axis at twice the roll frequency	$< 2 \times 10^{-7}$ g
5.	Root-mean-square acceleration	$< 2 \times 10^{-7}$ g
6.	Square root of the roll frequency component of the square of the acceleration	$< 10^{-8}$ g
7.	Square root of the twice roll frequency component of the square of the acceleration	$< 10^{-7}$ g
8.	Square root of the difference between the roll frequency components of the squares of the acceleration along two axes	$< 10^{-9}$ g

4. APPLICATION TO GROUND-BASED TESTING AND EVALUATION OF GYROSCOPES

Measurements of the drift rates of gyroscopes in a ground-based laboratory along with measurements of the physical characteristics of the gyroscopes is a useful method of verifying these results. For a gyroscope in a 1-g environment, which is spinning at 170 Hz, and which is spherical to within 10^{-6} centimeters, typical precession rates will be on the order of 0.1°/hr, which is significantly smaller than the 15°/hr rotation rate of the earth. Measurements of the drift rate due to the asphericities of the rotor will be most easily made if the rotor's spin axis is nearly aligned with the earth's rotation axis. In this configuration the drift rate due to the rotation of the earth is minimized and the orientation of the rotor's spin axis with respect to the local vertical does not change as the earth rotates. A facility for testing superconducting gyroscopes which are initially spun along the direction of the earth's rotation axis is now being constructed at Stanford.

Once these results for the torques on the gyroscope have been experimentally verified, the measured drift rate of gyroscopes may be used to determine the gyroscope's physical properties, such as mass unbalance or rotor asphericity. Only the components of the mass unbalance and rotor asphericity averaged over the rotation of the gyroscope contribute to the drift rate of the gyroscope. Since the orientation of the spin axis within the rotor is constantly changing due to the polhode motion, it might seem to be a hopeless task to extract information about the spin averaged quantites. However, by using the signals from the magnetic flux trapped in the rotor, it will be possible to measure the motion of the spin axis in the rotor and unravel the contributions to the rotor's drift rate. In addition, it will be possible to find some information about the differences in the moments of inertia of the rotor. The results of these measurements may be used as a method of qualifying gyroscopes for the flight experiment.

There are two interesting classes of gyroscope torques which are expected to be negligible in the flight experiment but which are observable in ground-based testing of gyroscopes: torques which change the rotor's spin speed and those which cause the polhode motion to be different than the polhode motion in the absence of external torques. If there is a variation in the voltage applied to the electrodes at the rotor spin-speed, then it is no longer possible to average over the rotation of the gyroscope and there will be some torques which change the rotor's spin speed. These torques must be carefully evaluated to find the correlation between the measured pressure in the housing and the spin-down rate of the gyroscope.

Those torques which affect the polhode motion of the rotor may be calculated using methods similar to those described in this paper but evaluating the torques in a rotor-fixed reference frame. At slow spin speeds the motion of the spin axis within the rotor is determined by these torques. As the spin speed increases the centrifugal torques which are responsible for the torque-free polhode motion become dominant. For ground-based testing of a flight-quality rotor the spin speed where the electrostatic torques are the same order of magnitude as the centrifugal torques is expected to be 12.5 Hz. For the same gyroscope in an orbiting drag-free satellite where the residual acceleration is less than 10^{-10} g, this critical spin speed becomes 1.25×10^{-4} Hz. For the flight experiment the gyroscope is to a very good approximation a free rigid rotor. For ground-based testing at the slower spin speeds these torques which change the polhode motion must be taken into account.

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