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Bianchi Type I Cosmological Models with Variable G and Λ: A Comment

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We treat in an alternate way a problem recently considered by Beesham [1]. We find that anisotropic Bianchi I inflationary cosmologies with variable gravitational and cosmological "constants" admit de Sitter expansion at least for late times.

In a recent paper, Beesham [1] has extended to the case of anisotropic Bianchi I models the work of Kalligas, Wesson and Everitt [2] on isotropic Friedmann–Robertson–Walker (FRW) models with time-variable gravitational and cosmological "constants". There are various reasons for considering the latter [3], and in recent years several papers have appeared wherein both parameters vary together in a way that leaves Einstein's equations formally unchanged [4–10]. We wish here to derive some results in Bianchi I cosmology with variable G and Λ using a slightly different method from that of Beesham [1], and to comment on his results.

We consider a Bianchi I universe

$$ds^2 = dt^2 - (\alpha_1(t)^2 dx_1^2 + \alpha_2(t)^2 dx_2^2 + \alpha_3(t)^2 dx_3^2) \tag{1}$$

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with a perfect-fluid energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{2}$$

and time-dependent G and Λ. Einstein's equations and the law of energy-momentum conservation ($T^{;\nu}_{\mu\nu} = 0$) after assuming an equation of state $p = w\rho$ (w = constant), read

$$\frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2} + \frac{\dot\alpha_1\dot\alpha_3}{\alpha_1\alpha_3} + \frac{\dot\alpha_2\dot\alpha_3}{\alpha_2\alpha_3} = 8\pi G\rho + \Lambda \tag{3}$$

$$\frac{\dot\alpha_1\dot\alpha_2}{\alpha_1\alpha_2} + \frac{\ddot\alpha_1}{\alpha_1} + \frac{\ddot\alpha_2}{\alpha_2} = -8\pi Gw\rho + \Lambda \tag{4}$$

$$\frac{\dot\alpha_1\dot\alpha_3}{\alpha_1\alpha_3} + \frac{\ddot\alpha_1}{\alpha_1} + \frac{\ddot\alpha_3}{\alpha_3} = -8\pi Gw\rho + \Lambda \tag{5}$$

$$\frac{\dot\alpha_2\dot\alpha_3}{\alpha_2\alpha_3} + \frac{\ddot\alpha_2}{\alpha_2} + \frac{\ddot\alpha_3}{\alpha_3} = -8\pi Gw\rho + \Lambda \tag{6}$$

$$\dot\rho + \rho(1+w)\left(\frac{\dot\alpha_1}{\alpha_1} + \frac{\dot\alpha_2}{\alpha_2} + \frac{\dot\alpha_3}{\alpha_3}\right) = 0. \tag{7}$$

Here a dot denotes differentiation with respect to t. It is useful to add eqs. (4), (5) and (6), and use (3) to obtain a relation without the first derivatives of the scale factors:

$$\frac{\ddot\alpha_1}{\alpha_1} + \frac{\ddot\alpha_2}{\alpha_2} + \frac{\ddot\alpha_3}{\alpha_3} = -4\rho(1+3w)\pi G + \Lambda. \tag{8}$$

The time derivatives of G and Λ are related by the Bianchi identities $(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu})^{;\nu} = 0 = (8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu})^{;\nu}$. It should be noted that this way of handling the conservation laws implies that G and Λ appear as indirectly coupled fields, similar to the case of G in the original Brans–Dicke theory [11]. In our case, we obtain

$$\dot\Lambda = -8\pi\dot G\rho. \tag{9}$$

Note that this equation is the same as in the isotropic case [2]. The system of eqs. (3), (7), (8) and (9) can be decoupled to obtain a single differential equation relating the energy density and the gravitational parameter following an analogous procedure with that in the isotropic models. We start from equations (7), and after squaring and using (3) we get

$$\left(\frac{\dot\rho}{\rho}\right)^2 = (1+\omega)^2\left[\left(\frac{\dot\alpha_1}{\alpha_1}\right)^2 + \left(\frac{\dot\alpha_2}{\alpha_2}\right)^2 + \left(\frac{\dot\alpha_3}{\alpha_3}\right)^2 + 16\pi G\rho + 2\Lambda\right]. \tag{10}$$

The time derivative of $\dot{\rho}/\rho$ from eq. (7) can now be expressed in terms of G,Λ and ρ only, by using eqs. (8) and (10). We obtain

$$\left(\frac{\dot{\rho}}{\rho}\right)^2 \frac{2+w}{(1+w)^2} - \frac{\ddot{\rho}}{\rho(1+w)} = 12\pi G\rho(1-w) + 3\Lambda\,. \tag{11}$$

Finally, on differentiating eq. (11) with respect to the time and using eq. (9), we can write

$$\rho(\dddot{\rho}) - \frac{5+3w}{1+w}\,\dot{\rho}\ddot{\rho} + \frac{4+2w}{1+w}\,\frac{\dot{\rho}^3}{\rho} = 12\pi\dot{G}\rho^3(1+w)^2 - 12\pi G\dot{\rho}\rho^2(1-w^2). \tag{12}$$

This is our central equation, and it corresponds to eq. (7) in our analysis of the isotropic models [2].

As with that equation, when trying to solve eq. (12), we can find analytic solutions only for a certain class of functions $G(t)$. At this point, it might be useful to stress that eq. (12) contains both the isotropic and the anisotropic sectors. To distinguish between the two, we use the anisotropy energy (σ):

$$8\pi G\sigma = \left(\frac{\dot{\alpha}_1}{\alpha_1} - \frac{\dot{\alpha}_2}{\alpha_2}\right)^2 + \left(\frac{\dot{\alpha}_1}{\alpha_1} - \frac{\dot{\alpha}_3}{\alpha_3}\right)^2 + \left(\frac{\dot{\alpha}_2}{\alpha_2} - \frac{\dot{\alpha}_3}{\alpha_3}\right)^2. \tag{13}$$

By expanding this and using eqs. (3) and (7), we can write

$$8\pi G\sigma = 18\left(\frac{\dot{R}}{R}\right)^2 - 48\pi G\rho - 6\Lambda = \frac{2}{(1+w)^2}\left(\frac{\dot{\rho}}{\rho}\right)^2 - 48\pi G\rho - 6\Lambda\,. \tag{14}$$

Here the second equation holds for $w \neq -1$ only, and we have made use of the average scale factor, $R(t) \equiv (\alpha_1\alpha_2\alpha_3)^{1/3}$, in terms of which eq. (7) reads

$$\dot{\rho} + 3(1+w)\frac{\dot{R}}{R}\rho = 0\,. \tag{15}$$

This and our preceding equations give back the isotropic solutions [2] in the appropriate limit.

However, we are mainly interested in finding solutions in the anisotropic sector $\sigma \neq 0$. This is not so straightforward, because eq. (11) even for constant G and zero Λ (standard Bianchi I cosmology) does not have analytic anisotropic solutions for every $w \neq -1$ [12]. Likewise in the present case, where G and Λ are allowed to vary with time, we have not been able to find such solutions even for simple functional forms of

$G(t)$. However, if one is willing to impose a relation between $G(t)$ and the equation of state constant w, then it is possible to find analytic anisotropic solutions for every $w \neq -1$. We proceed with identifying one class of solutions in this category. Thus consider

$$G = G(t, t_*) = Ct^{n_1}(t + t_*)^{n_2}, \tag{16}$$

where C,n_1,n_2,t_* are constants. The physical interpretation of t_* will be given below. (Note that one cannot eliminate t_* from eq. (16) by redefining the origin of time.) Now assume for the energy density

$$\rho(t, t_*) = At^{s_1}(t + t_*)^{s_2}. \tag{17}$$

After a straightforward but lengthy calculation, one finds that if we set

$$s_1 = -1 - w = s_2, \quad n_1 = n_2 = w, \quad A = \frac{s_1 s_2}{6\pi C(1 + w)^3}, \tag{18}$$

then (16) and (17) satisfy our central equation (12). Equation (9) when combined with (16), (17), and (18) gives

$$\dot{\Lambda} = -\frac{4w}{3(1 + w)}\left(\frac{1}{t^2(t + t_*)} + \frac{1}{t(t + t_*)^2}\right), \tag{19}$$

which integrates to

$$\Lambda = \frac{4w}{3(1 + w)}\,\frac{1}{t(t + t_*)}\,. \tag{20}$$

We see that the cosmological parameter preserves its natural $1/t^2$ behaviour for $t \gg t_*$. However, it varies as $1/t$ for $t \ll t_*$. The average scale factor can be obtained from (17) and (18) with the help of eq. (15). We find

$$R(t) = \text{const.} \times t^{1/3}(t + t_*)^{1/3}. \tag{21}$$

Finally, from the above expressions for G,ρ and Λ and eq. (14), we can write for the anisotropy energy

$$8\pi G\sigma = 2\left[\frac{t_*}{t(t + t_*)}\right]. \tag{22}$$

This equation implies that if $t_* = 0$ then $\sigma = 0$ too. Hence we can interpret t_* as the *isotropization time*. This is not surprising for if $w = 0$ then the above solutions reduce to the matter-dominated solutions of the

Bianchi I universe with constant G and vanishing Λ, where t_* has the same interpretation [13].

Let us now turn our attention to anisotropic solutions with constant density. It is easy to see from eq. (7) that $\dot{\rho} = 0$ is compatible either with a static universe ($\dot{\alpha}_i = 0,\ i = 1, 2, 3$) or with a vacuum equation of state ($w = -1$). Both cases have the corresponding FRW solutions [2] in the isotropic sector. In view of our definition of the anisotropy energy — eq. (13) — static solutions can only have $\sigma = 0$. Therefore, we concentrate on the case of $w = -1$, $\sigma \neq 0$. For a constant energy density ρ, eq. (9) can be immediately integrated to give

$$\Lambda(t) + 8\pi G(t)\rho = \text{constant} \equiv 3H_0^2. \tag{23}$$

This into eq. (14) yields

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2\left(1 + \frac{4\pi G\sigma}{9H_0^2}\right), \tag{24}$$

and if we set

$$\frac{4\pi G\sigma}{9H_0^2} = \sinh^{-2}(at), \qquad a = \text{const.}, \tag{25}$$

then (24) reads

$$\left(\frac{\dot{R}}{R}\right)^2 = H_0^2(1 + \sinh^{-2}(at)) = H_0^2 \operatorname{cotanh}^2(at). \tag{26}$$

The solution of this equation is

$$R(t) = \text{const.} \times \sinh^{H_0/a}(at). \tag{27}$$

At late times the above relation becomes $R(t) \propto \exp(H_0 t)$ which is the usual de Sitter expansion of the FRW models. Indeed, in this limit for nonzero $G(t)$, eq. (25) implies that σ decreases exponentially with time, and the universe isotropizes in the same way as in the usual Bianchi I cosmology with constant G and Λ [14]. Our work overlaps somewhat with the interesting recent paper by Beesham [1]. Our results complement his, and we see from (27) that inflationary (de Sitter) behaviour is allowed at least for late times.

To conclude, we have extended previous work by Kalligas, Wesson and Everitt [2] on isotropic cosmologies with variable G and Λ to anisotropic Bianchi I cosmologies. As before, we find that Λ varies as $1/t^2$ at late times, which matches its natural dimensions and may explain why the current value of this parameter is small. Our analysis has some similarities with recent work by Beesham [1], and we find that at least at late times de Sitter inflation is allowed.

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