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THE CLASSICAL TESTS IN KALUZA-KLEIN GRAVITY

D. KALLIGAS,[1,2,3] P. S. WESSON,[1,4,5] AND C. W. F. EVERITT[1,6]
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ABSTRACT

The possible existence of extra dimensions to spacetime can be tested astrophysically using Kaluza-Klein theory, which is a natural extension of Einstein's general relativity. In the simplest version of the theory, there is a standard class of five-dimensional solutions that are analogous to the four-dimensional Schwarzschild solution. However, even a small departure of the extra dimension from flatness affects the first or dominant part of the potential, making it possible to test for the existence of an extra dimension. Data from the solar system indicate that in our region of space the terms due to the fifth dimension are small ($\leq 0.1\%$) compared to those due to the usual four dimensions of spacetime. However, the parameters of Kaluza-Klein theory are not universal constants and can vary from place to place depending on local physics. Hence other astrophysical systems may serve as better laboratories for investigating the possible existence of extra dimensions.

Subject headings: cosmology: theory — gravitation — relativity — solar system: general

1. INTRODUCTION

This paper addresses the question of whether astrophysical observations might have any bearing on the possible existence of extra dimensions to spacetime. Physicists have been led to the idea that spacetime may have such extra dimensions by their attempts to unify all the fundamental interactions (e.g., Kaluza-Klein, supergravity, and superstring theories) (for reviews see the books by Lee 1984, De Sabbata & Schmutzer 1983; Applequist, Chodos, & Freund 1987; and Collins, Martin, & Squires 1989). It is often assumed that such extra dimensions could only be observed at energies above $\simeq 10^{14}$ GeV. We find that it may in principle be possible to test for their presence through macroscopic astrophysical observations.

There have been some attempts to work out astrophysical consequences of additional spacetime dimensions (Kostelecky & Samuel 1989a, b; Lim & Wesson 1992; Wesson 1992a, b; Coley, McManus, & Wesson 1994). But a systematic study of how to test for their existence has not hitherto been available. Therefore, we propose to take the basic form of Einstein's gravity for the simplest case of one extra spatial dimension, and work out all of the classical tests for it, along with some related effects.

When an extra dimension is added to general relativity, the theory becomes much richer in its physical consequences but also more complicated mathematically, so it is necessary to make some simplifying assumptions. Thus the metric is often taken to be independent of the extra dimension (i.e., there is an associated Killing vector). We likewise make this assumption, although without thereby restricting ourselves to a particular unification scheme. It is commonly assumed that the extra dimension is not amenable to direct observation. This may be because it has shrunk to inaccessible microscopic scales over cosmological time (Chodos & Detweiler 1980), because four-dimensional physics takes place on a constant hypersurface in a five-dimensional universe (Wesson 1992b), or because the extra dimension is rolled up or compactified to minute scales for reasons dictated by particle physics (Collins, Martin & Squires 1989). However, for generality we will not make specific assumptions about the nature of the extra dimension, even though for brevity we will continue to call the five-dimensional theory a Kaluza-Klein theory.

There is a canonical class of exact solutions for the one-body problem in five-dimensional general relativity that is analogous to the Schwarzschild solution in its four-dimensional counterpart. The latter is unique, by virtue of Birkhoff's theorem. Birkhoff's theorem, however, does not hold in five-dimensional general relativity with three-dimensional spherical symmetry. Consequently, there is a whole family of solutions characterized by two dimensionless parameters a and b. The Schwarzschild solution is recovered smoothly for $a \to 1$, $b \to 0$. If, however, b departs from zero then the object at the center of the geometry is a stable mass without an event horizon (Gross & Perry 1983; Davidson & Owen 1985; Dereli 1985; Chatterjee 1990; Liu 1991; Wesson 1992b). Thus if the world is five-dimensional, the metric outside the Sun belongs to this more general category, and one can work out the classical tests for it, compare them to observations, and thereby constrain the existence and nature of the fifth dimension. One may wonder why an extra dimension could have observable effects in celestial mechanics even if it is confined to a microscopic scale. The reason is that the constant b which defines the nature of the fifth dimension is coupled by a consistency relation to the constant a which defines the dominant or time part of the metric (see below). Therefore, a departure of b from zero results in a departure of a from unity, which affects the dominant potential and alters the classical tests. In this way, the motions of photons and planets can significantly constrain the presence of an extra dimension in the solar systen.

In other stellar systems, the dimensionless parameter b could be different from what it is in our own system. There is no known way to fix b a priori (it is not a universal constant), although it is commonly assumed to be related to the density and internal

[1] GP-B, HEPL, Stanford University, Stanford, CA 94305.
[2] Department of Applied Physics, Stanford University, Stanford, CA 94305.
[3] dimitri@relgyro.stanford.edu.
[4] Department of Physics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
[5] Wesson@watsci.uwaterloo.ca.
[6] francis@relgyro.stanford.edu.

pressure of the object at the center of the geometry (Gross & Perry 1983; Davidson & Owen 1985; Wesson 1992a). In searching for evidence of extra dimensions, therefore, it is worthwhile examining data from systems such as multiple stars (like DI Herculis), binary pulsars and extrasolar planetary systems.

In anticipation of good data becoming available for systems other than our own, we have kept the following work general in scope. Section 2 deals with photons, including the light deflection experiment (2.1), radar ranging (2.2), and the possible existence of circular orbits for light around central masses (2.3). Section 3 deals with massive test particles, including the perihelion advance of a planet (3.1), and the possible radial infall of a particle toward a central mass (3.2). Section 4 provides a discussion of the redshift effect (4.1) and the geodetic precession for a spinning particle in orbit around a central body (4.2). Section 5 is a conclusion. Some relations that are necessary but of secondary importance are given in the Appendix.

2.1. *Deflection of Light*

We start by examining the motion of massless particles in five-dimensional spacetime with the Gross-Perry line element

$$ds^2 = A^a\,dt^2 - A^{-(a+b)}\,dr^2 - A^{(1-a-b)}r^2(d\theta^2 + \sin^2\theta\,d\phi^2) - A^b\,dn^2\,, \tag{1}$$

with

$$A(r) \equiv (1 - 2M/r)\,.$$

together with the consistency condition $1 = a^2 + ab + b^2$. Here, M is a parameter related to the gravitational mass of the central body (see § 4.1).

The Lagrangian for particles with zero mass in the metric of equation (1),

$$L = A^a\dot{t}^2 - A^{-(a+b)}\dot{r}^2 - A^{(1-a-b)}r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - A^b\dot{n}^2 \tag{2}$$

has the same form as that for massive particles, only now an overdot denotes the derivative with respect to an affine parameter (say λ) along the null geodesics. (Strictly speaking eq. [2] is the square of the Lagrangian density, but since it leads to the same equations of motion, hereafter we will use it and refer to it, for brevity, as the Lagrangian.) Without loss of generality we confine ourselves to orbits with $\theta = \pi/2$, $\dot{\theta} = 0$. From equation (2) we can see that n, ϕ, t are cyclic coordinates, so we have three constants of the motion:

$$l \equiv A^a\dot{t} = \text{constant}\,, \tag{3}$$

$$h \equiv A^{(1-a-b)}r^2\dot{\phi} = \text{constant}\,, \tag{4}$$

$$k \equiv A^b\dot{n} = \text{constant}\,. \tag{5}$$

The line element vanishes for null geodesics, $ds^2 = 0$. This into equation (1) yields, after dividing by $\dot{\phi}^2$,

$$A^a\,\frac{\dot{t}^2}{\dot{\phi}^2} - A^{-(a+b)}\,\frac{\dot{r}^2}{\dot{\phi}^2} - A^{(1-a-b)}r^2 - A^b\,\frac{\dot{n}^2}{\dot{\phi}^2} = 0\,. \tag{6}$$

By making use of our constants of motion from equations (3)–(5) we can rewrite equation (6) in a form containing only $\dot{r}^2/\dot{\phi}^2 = (dr/d\phi)^2$. We obtain

$$\left(\frac{dr}{d\phi}\right)^2 - (A^{(2-2a-b)}l^2 - A^{(2-a-2b)}k^2)\,\frac{r^4}{h^2} + Ar^2 = 0\,. \tag{7}$$

Rewriting in terms of $u \equiv 1/r$, equation (7) takes the form

$$\left(\frac{du}{d\phi}\right)^2 - [A^{(2-2a-b)}l^2 - A^{(2-a-2b)}k^2]\,\frac{1}{h^2} + Au^2 = 0\,. \tag{8}$$

In the weak field limit

$$Mu = M/r \ll 1\,, \tag{9}$$

equation (8) reads

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = F + \epsilon pu + \epsilon u^3\,. \tag{10}$$

Here we have written $\epsilon = 2M$, and F, p are constants given by

$$F \equiv \frac{l^2}{h^2} - \frac{k^2}{h^2}\,, \tag{11}$$

$$p \equiv (2 - a - 2b)\,\frac{k^2}{h^2} - (2 - 2a - b)\,\frac{l^2}{h^2}\,. \tag{12}$$

We are interested in the deviation from a straight path for a photon, coming from plus infinity and escaping to minus infinity due to the presence of a central body. At the point of closest approach, u_0, we have $du/d\phi = 0$; this into equation (10) yields

$$F = u_0^2(1 - \epsilon u_0 - \epsilon p/u_0) . \tag{13}$$

By inserting this into equation (10) we obtain

$$\left(\frac{du}{d\phi}\right)^2 + u^2 = u_0^2[1 - \epsilon u_0 - (\epsilon p/u_0)] + \epsilon pu + \epsilon u^3 . \tag{14}$$

As we consider the weak field regime, ϵ is a small quantity. Working to first order, we find for u

$$u(\phi) = u_0 \sin\phi + \frac{\epsilon u_0^2}{2}(1 + \cos^2\phi - \sin\phi) + \frac{\epsilon p}{2}(1 - \sin\phi) + \epsilon\left(u_0^2 + \frac{p}{2}\right)\cos\phi , \tag{15}$$

or equivalently,

$$u(\phi) = \left[u_0\left(1 - \frac{\epsilon u_0}{2}\right) - \frac{\epsilon p}{2}\right]\sin\phi + \frac{\epsilon u_0^2}{2}(1 - \cos\phi)^2 + \frac{\epsilon p}{2}(1 - \cos\phi) . \tag{16}$$

In the limit where ϵ goes to zero, equation (16) gives

$$u(\phi) = u_0 \sin\phi , \tag{17}$$

which is a straight line for the path of a photon originating from $+\infty$ at $\phi = 0$ and escaping to $-\infty$ along the direction $\phi = \pi$. For nonzero ϵ the photon will deflect and escape to $-\infty$ along the direction $\phi = \pi + \omega$. In view of equation (9) we expect ω to be small, therefore we will only keep terms linear in ϵ and ω in the following. We can then find ω, with the help of equation (16), by demanding $u(\phi = \pi + \omega) = 0$; we obtain

$$\omega = \frac{4M}{r_o} + 2Mpr_o . \tag{18.1}$$

The first term in equation (18.1) is the familiar result of four-dimensional general relativity. The deviation from this result dictated by the second term has a somewhat unusual form, as it might appear to imply that the deflection angle would grow with the impact parameter r_o. However, as p involves the "angular momentum" constant h in the denominator (see eq. [12]), it will be proportional to $1/r_o^2$, so $2Mpr_o$ will also diminish with increasing r_o. It can be seen from equations (12) and (18.1) that $2Mpr_o$ does not vanish even if the photon has no velocity component in the fifth dimension, unless we also have $a = 1$, $b = 0$, which is the Schwarzschild limit. To demonstrate this insert $k = 0$ into equation (12); then from equations (3), (4), and (18.1) we can write for the deflection angle

$$\omega = \frac{(4a + 2b)M}{r_o} , \tag{18.2}$$

which clearly reproduces the usual general relativity result when $a = 1$ and $b = 0$.

As an illustration, let us consider a photon passing the Sun with impact parameter $r_o = R_\odot$, and assume $b = 7.5 \times 10^{-3}$. Then the consistency relation gives $a = 0.9962289$. If for simplicity we also take $k = 0$, the values of the second and first terms in (18.1) are found to be in the ratio 3.76×10^{-3}, which represents a small decrease in the deflection angle. In terms of the reported accuracy for this type of experiment (Will 1992), such a departure would be marginally detectable if the Sun had the noted value of b. Clearly a greater deviation from the Schwarzschild value of $b = 0$ would produce greater effects in equation (18) that are in principle measureable.

The relations (18) allow not only for light attraction but also for *light repulsion* for certain values of the parameters (e.g., $a = 0.5$, $b = -1.1513878$, $k = 0$). There is actually a whole range of the parameter a for which light repulsion occurs. Furthermore, there also exists the possibility for *null deflection* [when $k = 0$, this happens for $a = \pm 1/(3)^{1/2}$, $b = \mp 2/(3)^{1/2}$, only]. These possibilities demonstrate the peculiar consequences that the presence of an extra dimension may lead to.

2.2. *Radar Ranging*

The presence of spacetime curvature leads to an increase in the travel time of light between any two given points as compared to its flat space value. Measuring this increase in travel time using the propagation of radar signals in our solar system was first proposed by Shapiro in 1964, and it constitutes the "fourth test" of general relativity, also known as *radar ranging* (see Will 1992). Here we will calculate the time delay for radar ranging in the spacetime of equation (1).

The usual scenario has an observer on Earth emitting radar signals which are reflected back by another planet. By choosing our coordinates so that this planet, Earth, and the Sun lie on the $\theta = \pi/2$ plane, then to a first approximation we can use equation (17) for the path of the light signal (the actual curvature of the path makes a negligible contribution to the time delay.) By differentiating equation (17) we obtain

$$r^2\,d\phi^2 = \frac{r_o^2}{r^2 - r_o^2}\,dr^2 , \tag{19}$$

r_o being the distance of closest approach to the Sun. If we now multiply equation (6) by $\dot{\phi}^2/\dot{t}^2$ and use equation (19) together with equations (3) and (5) we find

$$A^a = A^{-(a+b)}\left(\frac{dr}{dt}\right)^2 + A^{1-a-b}\frac{r_o^2}{r^2 - r_o^2}\left(\frac{dr}{dt}\right)^2 + A^{2a-b}\left(\frac{k}{l}\right)^2 , \tag{20}$$

which can be solved for

$$\frac{dr}{dt} = \left\{\left[1 - A^{a-b}\left(\frac{k}{l}\right)^2\right]\Big/\left[A^{-(2a+b)} + A^{1-2a-b}\frac{r_o^2}{r^2 - r_o^2}\right]\right\}^{1/2} . \tag{21}$$

The coordinate time for the roundtrip can then be found by integrating equation (21):

$$\begin{aligned}(\Delta t)_{\text{roundtrip}} = 2\Bigg[\int_{r_o}^{r_p} &\left\{\left[A^{-(2a+b)} + A^{1-2a-b}\frac{r_o^2}{r^2 - r_o^2}\right]\Big/\left[1 - A^{a-b}\left(\frac{k}{l}\right)^2\right]\right\}^{1/2} dr \\ &+ \int_{r_o}^{r_e}\left\{A^{-(2a+b)} + A^{1-2a-b}\frac{r_o^2}{r^2 - r_o^2}\Big/\left[1 - A^{a-b}\left(\frac{k}{l}\right)^2\right]\right\}^{1/2} dr\Bigg] .\end{aligned} \tag{22}$$

In the weak field regime of equation (9) the integrand in equation (22) can be written in the form,

$$\frac{r}{\sqrt{r^2 - r_o^2}}\left\{1 + \frac{1}{2}\left(\frac{k}{l}\right)^2 + \left[(2a + b) + \frac{3b}{2}\left(\frac{k}{l}\right)^2\right]\frac{M}{r} - \left[1 + \left(\frac{k}{l}\right)^2\right]\frac{Mr_o^2}{r^3}\right\} . \tag{23}$$

This into equation (22) gives

$$\begin{aligned}(\Delta t)_{\text{roundtrip}} = 2\Bigg\{&\left[1 + \frac{1}{2}\left(\frac{k}{l}\right)^2\right]\left(\sqrt{r_p^2 - r_o^2} + \sqrt{r_e^2 - r_o^2}\right) + M\left[(2a + b) + \frac{3b}{2}\left(\frac{k}{l}\right)^2\right] \\ &\times\left[\ln\left(\frac{r_p + \sqrt{r_p^2 - r_o^2}}{r_o}\right) + \ln\left(\frac{r_e + \sqrt{r_e^2 - r_o^2}}{r_o}\right)\right] - M\left[1 + \frac{1}{2}\left(\frac{k}{l}\right)^2\right]\left(\frac{\sqrt{r_p^2 - r_o^2}}{r_p} + \frac{\sqrt{r_e^2 - r_o^2}}{r_e}\right)\Bigg\} ,\end{aligned} \tag{24}$$

where r_p and r_e denote the radius measures to the planet (reflector) and Earth (emitter) correspondingly. The proper time for the trip will depend on the position of measurement. This usually is $r = r_e$, so (see also § 4.1),

$$(\Delta s)_{\text{roundtrip}} = \left(1 - \frac{2M}{r}\right)^a (\Delta t)_{\text{roundtrip}} \simeq \left(1 - \frac{2aM}{r}\right)(\Delta t)_{\text{roundtrip}} . \tag{25}$$

For $a = 1$, $b = 0$, and $k = 0$, equation (24) gives back the result of usual 4D general relativity. Any deviation from these values of a, b, and k for our solar system must be such that the resulting fractional change in $(\Delta t)_{\text{roundtrip}}$ be less than 0.1% (Will 1992). For $k = 0$, the prefactor of the logarithmic terms in equation (24) becomes negative or null, for the same values of a and b as in the light deflection problem.

2.3. *Photon Circular Orbits*

In Schwarzschild spacetime, photons can be set in circular motion around a central object of mass M for a particular value of the radial coordinate, which happens to be $r = 3M$. Here, we examine how this is different in the five-dimensional metric of equation (1). In view of equation (2) the Lagrange equation with $\dot{r}$ is

$$\begin{aligned}&-2A^{-(a+b)}\ddot{r} + 2(a + b)A^{-(a+b+1)}A_{,r}\dot{r}^2 - aA^{a-1}A_{,r}\dot{t}^2 - (a + b)A^{-(a+b+1)}A_{,r}\dot{r}^2 \\ &\quad + (1 - a - b)A^{-(a+b)}A_{,r}r^2\dot{\phi}^2 + 2A^{(1-a-b)}r\dot{\phi}^2 + bA^{(b-1)}A_{,r}\dot{n}^2 = 0 .\end{aligned} \tag{26}$$

For circular orbits r will be a constant; therefore the above equation simplifies to

$$-aA^{a-1}A_{,r}\dot{t}^2 + (1 - a - b)A^{-(a+b)}A_{,r}r^2\dot{\phi}^2 + 2A^{(1-a-b)}r\dot{\phi}^2 + bA^{(b-1)}A_{,r}\dot{n}^2 = 0 . \tag{27}$$

If we divide this equation by $\dot{\phi}^2$, and use $A_{,r} = 2M/r^2$ and $\dot{n}^2/\dot{\phi}^2 = k^2A^{(2-2a-4b)}r^4/h^2$ from eqs. [3] and [5]), we obtain

$$\frac{\dot{t}^2}{\dot{\phi}^2} = \frac{(1 - a - b)}{a}A^{(1-2a-b)}r^2 + A^{(2-2a-b)}\frac{r^3}{Ma} + A^{(2-3a-3b)}\frac{b^2k^2r^4}{ah^2} . \tag{28}$$

From equation (6), however, and after setting $\dot{r} = 0$ we find another expression for $\dot{t}^2/\dot{\phi}^2$, namely:

$$\frac{\dot{t}^2}{\dot{\phi}^2} = A^{(1-2a-b)}r^2 + \frac{k^2}{h^2}A^{(2-3a-3b)}r^4 . \tag{29}$$

For consistency, equations (28) and (29) yield

$$(1 - 2a - b)r^2 + \frac{A}{M}r^3 + (b - a)A^{(1-a-2b)}\frac{k^2r^4}{h^2} = 0 . \tag{30}$$

The solutions to this equation give the permitted radii for photon circular orbits. For the case of negligible motion along the fifth dimension we obtain

$$r = (1 + 2a + b)M . \tag{31}$$

In the Schwarzschild limit ($a = 1$, $b = 0$) this consistently reduces to $r = 3M$. On the other hand, for $a = 1$ and $b = -1$ (a case studied independently by Chatterjee 1990) we have $r = 2M$. In this special case photons can only be put into circular orbit at a distance from the center equal to the horizon radius of the 4D Schwarzschild solution.

3.1. *Perihelion Advance*

The potential use of the Einstein perihelion advance as a probe for extra dimensions has been examined in earlier work (Lim & Wesson 1992). In view of this we will only summarize the calculation and present the results. The reader is referred to the noted paper for further details.

The motion of massive particles can be obtained from the Lagrangian of equation (2) where overdots denote the derivatives with respect to the proper time s. With this change in notation, the three conservation laws (3), (4), and (5) remain intact. For massive particles the right-hand side of equation (6) becomes $1/\phi^2$, and as a result equation (8) reads instead:

$$\left(\frac{du}{d\phi}\right)^2 - (-A^{(2-a-b)} + A^{(2-2a-b)}l^2 - A^{(2-a-2b)}k^2)\frac{1}{h^2} + Au^2 = 0 . \tag{32}$$

We now differentiate this equation with respect to ϕ to obtain

$$u'u'' + uu' = \frac{Mu'}{h^2}[(2 - a - b)A^{(1-a-b)} + (-2 + 2a + b)A^{(1-2a-b)}l^2 + (2 - a - 2b)A^{(1-a-2b)}k^2] + 3Mu^2u' , \tag{33}$$

with a prime denoting differentiation with respect to ϕ. Concentrating on noncircular orbits $u' \neq 0$, we can eliminate u' from equation (33). If we further consider the weak-field regime (see eq. [9]), we can write after neglecting terms of order $(Mu)^2$ and higher:

$$u'' + (1 + \gamma\epsilon)u = B + \frac{\epsilon u^2}{B} , \tag{34}$$

where

$$\gamma \equiv -\frac{e}{3d} , \quad \epsilon \equiv \frac{3M^2 d}{h^2} \ll 1 , \quad B \equiv \frac{Md}{h^2} , \quad d \equiv (1 + k^2) + (a - 1)(-1 + 2l^2 - k^2) + b(-1 + l^2 - 2k^2) , \tag{35}$$

$$e \equiv 2(2 - a - b)(-1 + a + b) + 2l^2(-2 + 2a + b)(-1 + 2a + b) + 2k^2(2 - a - 2b)(-1 + a + 2b) . \tag{36}$$

The solution to equation (34) to first order in ϵ is

$$u = \frac{1}{r} = B + \epsilon B(1 - \gamma) + \epsilon\frac{C^2}{2B} - \epsilon\frac{C^2}{6B}\cos 2\phi + \left(1 - \frac{\gamma}{2}\right)C \cos\left\{\left[1 - \epsilon\left(1 - \frac{\gamma}{2}\right)\right]\phi\right\} , \tag{37}$$

where C is an integration constant. The perihelion shift arises from the lack in periodicity exhibited by equation (37). Its value between successive perihelia amounts to

$$\delta\phi = 2\pi\epsilon\left(1 - \frac{\gamma}{2}\right) = \frac{6\pi M^2}{h^2}\left(d + \frac{e}{6}\right) . \tag{38.1}$$

For $a = 1$, $b = 0$ and $k = 0$, it is evident from equations (35) and (36) that $d = 1$ and $e = 0$, thus we obtain the result of the standard 4D theory (Adler, Basin, & Schiffer 1975). For a nearly circular orbit, equation (38.1) takes the form (cf. § A3),

$$\delta\phi = \frac{6\pi M}{r}\left(a + \frac{2}{3}b\right) , \tag{38.2}$$

where r is the orbit's coordinate radius.

For nonzero k, as equation (38.1) indicates, the expression for $\delta\phi$ is no longer clear cut, unless one can separate the effects of the "potentials" (a, b) from the velocity in the fifth dimension (k). The latter is most easily isolated by looking at the 3D-effective motion of a particle, especially radially, and this we now proceed to examine.

3.2. *Radial Free-Fall of Massive Particles*

One of the profound differences between Schwarzschild spacetime and the space of classical Newtonian theory is found when examining the vertical free-fall of a test particle (with mass m) toward a central body of mass M. The classical theory has the particle's speed constantly increasing from its initial value all the way to infinity as r tends to zero. Contrary to this, in the Schwarzschild metric the particle's coordinate speed $u_r \equiv |(dr/dt)|$ will start decreasing at $r = 6M$ and reach the Schwarzschild surface $r = 2M$ with $u_r = 0$. In this section we will examine the vertical free-fall of massive test particles in the spacetime of equation (1) and see how these results are modified.

For vertical free-fall we have $d\theta = 0 = d\phi$, and by inserting this into equation (1) and dividing by ds^2 we obtain

$$A^a\dot{t}^2 - A^{-(a+b)}\dot{r}^2 - A^b\dot{n}^2 = 1\,, \tag{39}$$

where an overdot here denotes the derivative with respect to s. Using the relations (3) and (5) we can rewrite this in the form

$$\dot{r}^2 + A^a k^2 - A^b l^2 + A^{(a+b)} = 0\,. \tag{40}$$

For a particle at rest ($\dot{r} = 0$) at $r = r_o$ the above relation gives

$$l^2 = [A(r_o)]^{(a-b)}k^2 + [A(r_o)]^a\,. \tag{41}$$

Equation (40) into equation (41) yields

$$\dot{r}^2 = \left[\left(1 - \frac{2M}{r}\right)^b\left(1 - \frac{2M}{r_o}\right)^{a-b} - \left(1 - \frac{2M}{r}\right)^a\right]k^2 + \left(1 - \frac{2M}{r}\right)^b\left(1 - \frac{2M}{r_o}\right)^a - \left(1 - \frac{2M}{r}\right)^{a+b}, \tag{42}$$

which can be thought of as an "energy equation" for our problem. Indeed, it is easily seen that in the limit where $a = 1$, $b = 0$, and $k = 0$ it reduces to $\dot{r}^2/2 = M(1/r - 1/r_o)$, which is the corresponding expression for the Schwarzschild metric and has the same form as the energy equation (for vertical free-fall) in classical Newtonian theory.

Now for our problem with five-dimensional interval, the coordinate velocity component in the r-direction is

$$\frac{dr}{dt} = \frac{dr}{ds}\frac{ds}{dt}\,, \tag{43}$$

and from equation (3) we have $dt/ds = A^{-a}l$, so by using expression (41) for l we can write

$$\frac{ds}{dt} = \left(1 - \frac{2M}{r}\right)^a \Bigg/ \sqrt{\left(1 - \frac{2M}{r_o}\right)^{a-b}k^2 + \left(1 - \frac{2M}{r_o}\right)^a} \tag{44}$$

Furthermore, equation (42) yields

$$dr/ds = -\{[(1 - 2M/r)^b(1 - 2M/r_o)^{a-b} - (1 - 2M/r)^a]k^2 + (1 - 2M/r)^b(1 - 2M/r_o)^a - (1 - 2M/r)^{a+b}\}^{1/2}\,, \tag{45}$$

which when combined with equation (44) gives

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)^a\Bigg\{\left[\left(1 - \frac{2M}{r}\right)^b\left(1 - \frac{2M}{r_o}\right)^{a-b} - \left(1 - \frac{2M}{r}\right)^a\right]k^2 + \left(1 - \frac{2M}{r}\right)^b\left(1 - \frac{2M}{r_o}\right)^a - \left(1 - \frac{2M}{r}\right)^{a+b}\Bigg\}^{1/2} \Bigg/ \left[\left(1 - \frac{2M}{r_o}\right)^{a-b}k^2 + \left(1 - \frac{2M}{r_o}\right)^a\right]^{1/2} \tag{46}$$

For a particle starting at an infinite radial distance from the central body, equation (46) reduces to

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)^a\left\{\left[\left(1 - \frac{2M}{r}\right)^b - \left(1 - \frac{2M}{r}\right)^a\right]k^2 + \left(1 - \frac{2M}{r}\right)^b\left[1 - \left(1 - \frac{2M}{r}\right)^a\right]\right\}^{1/2} \Bigg/ \sqrt{1 + k^2}\,. \tag{47}$$

One feature of this result is that the radial speed of a test particle will depend on its velocity component along the fifth dimension (through k). In situations where the latter cannot be neglected, an effectively four-dimensional (distant) observer will interpret this dependence as an *apparent violation of the equivalence principle*. This is because different test particles will have different values of k and therefore different values of dr/dt. (It is easy to see, after multiplying equation (47) by $(g_{rr})^{1/2}/(g_{tt})^{1/2} = A^{-(a+b/2)}$ that a similar statement holds true for the locally measured radial speed.) For cases of negligible k a straightforward, yet rather lengthy, calculation shows that the radius where dr/dt starts decreasing is given by

$$r^* = 2M \Bigg/ \left[1 - \left(\frac{2a + b}{3a + b}\right)^{1/a}\right] \tag{48}$$

In the limit, $a = 1$, $b = 0$ equation (34) consistently reduces to $r^* = 6M$, the general relativity value, as expected. For the choice of parameters discussed by Chatterjee (1990), r^* is reduced to $r^* = 4M$.

4.1. *Frequency Shift*

To investigate the gravitational redshift formula in the spacetime of equation (1) we note that the latter is a static spacetime, so we can consider emitters and receivers of light signals with fixed spatial coordinates. We then write as usual

$$\frac{\nu_r}{\nu_e} = \frac{g_{00}(x_e^a)}{g_{00}(x_r^a)}\,, \tag{49}$$

for the ratio of the frequency of a signal emitted at $x_e^a = (r_e, \theta_e, \phi_e, n_e)$ and received at $x_r^a = (r_r, \theta_r, \phi_r, n_r)$. Equation (49) with the help of equation (1) becomes

$$\frac{\nu_r}{\nu_e} = \left[\left(1 - \frac{2M}{r_e}\right)\Bigg/\left(1 - \frac{2M}{r_r}\right)\right]^{a/2} \simeq 1 + aM\left(\frac{1}{r_r} - \frac{1}{r_e}\right) + O(M^2)\,, \tag{50}$$

where the second equation refers to the weak field regime of equation (9). In view of equation (50) we can write

$$\frac{v_r - v_e}{v_e} = aM\left(\frac{1}{r_r} - \frac{1}{r_e}\right). \tag{51}$$

This is to be compared to the result of four-dimensional general relativity:

$$\frac{v_r - v_e}{v_e} = M\left(\frac{1}{r_r} - \frac{1}{r_e}\right). \tag{52}$$

Equations (51) and (52) will be in agreement if we identify the gravitational mass of the central body in the spacetime of equation (1) via

$$M_{\text{grav}} = aM . \tag{53}$$

From this equation we can see why Chatterjee's choice of parameters stands out, as it leads to $M_{\text{grav}} = M$ as in the Schwarzschild solution, while still allowing for a curved fifth dimension.

4.2. *Geodetic Effect*

Consider a spacelike vector S^M and a timelike circular orbit, $x^A = x^A(s)$, in the spacetime of equation (1). We will compute the change in the spatial orientation of S^M after parallel transport once around the circular orbit. The requirement of parallel transport implies that

$$\frac{dS^M}{ds} + \Gamma^M_{AB} S^A \dot{x}^B = 0 , \tag{54}$$

where an overdot denotes differentiation with respect to the proper time s. Since S^M is spacelike and $\dot{x}^A$ is timelike, their inner product can be made to vanish:

$$g_{MN} S^M \dot{x}^N = 0 . \tag{55}$$

Our objective is to solve equations (54) and (55) for the S^M as functions of proper time s. In order to simplify the calculation we will assume that $S^5 = 0$. Since $x^A = x^A(s)$ is a circular orbit, we can write for $\dot{x}^A$

$$\dot{x}^A(s) = \left(\dot{t}, 0, 0, \frac{d\phi}{dt}\,\dot{t}, 0\right). \tag{56}$$

The expressions for $\dot{t}$ and $\Omega \equiv d\phi/dt$ are found in the Appendix to be

$$\dot{t} = \left(\frac{1-2M}{r}\right)^{-a/2} \Big/ \left[1 - \frac{aM}{r-(1+a+b)M}\right]^{-(1/2)} \tag{57}$$

and

$$\Omega = \left[\frac{(1-a-b)}{a}\left(\frac{1-2M}{r}\right)^{1-2a-b} r^2 + \frac{1}{aM}\left(\frac{1-2M}{r}\right)^{2-2a-b} r^3\right]^{1/2} . \tag{58}$$

Equation (56) into equation (55) gives

$$g_{00} S^0 \dot{x}^0 + g_{33} S^3 \dot{x}^3 = 0 , \tag{59}$$

which in view of equation (1) becomes

$$S^0 = (1 - 2M/r)^{1-2a-b} \Omega r^2 S^3 . \tag{60}$$

This relation between S^0 and S^3 makes the system of equations (54) redundant since the $M = 0$ and $M = 3$ equations are the same. Furthermore, based on our assumption that $S^5 = 0$, together with equation (56) and our expressions for the nonzero Christoffel symbols, the $M = 5$ equation in equation (54) becomes trivial. Hence we are left with three out of the five equations (54). In terms of the nonzero Γ's these are

$$\frac{dS^1}{ds} + \Gamma^1_{00} S^0 \dot{x}^0 + \Gamma^1_{33} S^3 \dot{x}^3 = 0 , \tag{61a}$$

$$\frac{dS^2}{ds} = 0 , \tag{61b}$$

$$\frac{dS^3}{ds} + \Gamma^3_{13} S^1 \dot{x}^3 = 0 . \tag{61c}$$

With the help of equations (56) and (A1) the above system becomes

$$\frac{dS^1}{ds} - \frac{r-(1+a+b)M}{(r-2M)^a} \frac{\Omega r^a}{\dot{t}} S^3 = 0 , \tag{62a}$$

$$S^2 = \text{constant} , \tag{62b}$$

$$\frac{dS^3}{ds} + \frac{r-(1+a+b)M}{r(r-2M)} \Omega \dot{t} S^1 = 0 . \tag{62c}$$

The solution of the system (62) is

$$S^1 = \frac{H}{r}\left\{[r(r-2M)]\,\frac{r-(1+2a+b)M}{r-(1+a+b)M}\right\}^{1/2}$$

$$\times \cos\left\{\phi_0 - r^{(a-1)/2}[r-(1+a+b)M](r-2M)^{-(a+1)/2}s\Bigg/\left[\frac{(1-a-b)}{a}A^{1-2a-b}r^2 + A^{2-2a-b}\frac{r^3}{Ma}\right]^{1/2}\right\}, \tag{63a}$$

$$S^2 = \text{constant}, \tag{63b}$$

$$S^3 = \frac{H}{r}\sin\left\{\phi_0 - r^{(a-1)/2}[r-(1+a+b)M](r-2M)^{-(a+1)/2}s\Bigg/\left[\frac{(1-a-b)}{a}A^{1-2a-b}r^2 + A^{2-2a-b}\frac{r^3}{Ma}\right]^{1/2}\right\}. \tag{63c}$$

where H and ϕ_0 are constants. We can see from equations (58) and (63) that the spatial part of S^M rotates relative to the radial direction with proper angular speed Ω_S, given by

$$\Omega_S = [r^{(a-1)/2}[r-(1+a+b)M]/(r-2M)^{(a+1)/2}]\Omega\,. \tag{64}$$

In usual general relativity we have $\Omega_S = \Omega$. It is easily seen that equation (64) is in agreement with this result in the appropriate limit $a \to 1, b \to 0$.

If S^M was originally radially oriented, then, upon completion of one revolution along $x^A(s)$ its direction will be shifted from the radial by

$$\delta\phi = 2\pi\left[1 - \frac{\sqrt{r-(1+a+b)M}\sqrt{r-(1+2a+b)M}}{\sqrt{r(r-2M)}}\right]. \tag{65}$$

In the weak-field limit this expression reduces to

$$\delta\phi = \frac{(3a+2b)\pi M}{r}\,. \tag{66}$$

Consider then a gyroscope in a 650 km altitude orbit around Earth, as in the Gravity-Probe B mission. Assume that a, b for Earth have identical values with the a, b used in the light deflection calculation of § 2.1, viz., $b = 7.5 \times 10^{-3}$, $a = 0.9962289$. Then equation (66) gives $\delta\phi \simeq 1.230$ mas per revolution, or an angular rate of 6".633 yr^{-1}. This is to be compared to the usual general relativity value of 6".625 yr^{-1} (obtained also from eq. [66] for $a = 1$ and $b = 0$; this angle comes from closing the orbit in a geodetic advance of the spin vector.) Since Gravity-Probe B is expected to measure angular rates approaching 0.1 mas yr^{-1} it should easily be able to detect such a difference.

As with light deflection and radar ranging, particular ranges of a, b may yield a negative effect (e.g., $a = 0.6$, $b = -1.1544$; $k = 0$), i.e., *a geodetic regression* instead of a geodetic advance of the spin vector. For ($a = \pm 2/(7)^{1/2}$, $b = \mp 3/(7)^{1/2}$; $k = 0$) there is no geodetic precession. Notice that these values are different than those corresponding to null light deflection. Hence, in principle we can have systems with zero light deflection and radar delay which exhibit nonzero geodetic precession or vice versa. We can also have systems with positive (negative) light deflection and radar ranging effect which exhibit negative (positive) geodetic effect.

We have derived the classical tests and some related effects in a Kaluza-Klein theory where the spacetime of Einstein's general relativity is extended by one extra dimension. The indirect effects of the extra dimension can be examined using a class of exact solutions which are spherically symmetric in ordinary three-dimensional space and specified by equation (1). In the special case where the extra dimension is flat, this class of solutions reduces to the usual Schwarzschild solution. In general, however, these solutions depend on two parameters which are not universal constants and depend on the local physics. There is a consistency relation between these two constants, which means that even a small departure of the fifth dimension from flatness will affect the first or dominant part of the metric. Thus even if an extra dimension is microscopic and not directly visible, it can be tested for.

In this regard, the deflection of light from the Sun shows new effects, including the possibility of *repulsion.* The radar ranging (time delay) test is likewise modified. For a massive test particle, the perihelion advance of an elliptical orbit and the radial infall case also show new terms not present in the usual four-dimensional theory. The geodetic precession of a test object in orbit around a central mass is also modified, and for a certain range of the parameters there can be geodetic regression. We thus have a set of new equations which contain terms dependent on the presence of the fifth dimension and can be compared to observation.

Existing data show that within our solar system the new terms must be relatively small ($\lesssim$0.1%; Will 1992). However, this does not constitute a universal constraint. It is actually a generic property of Kaluza-Klein theory that the parameters which appear in the solutions (1) are different in different systems, with values that depend on the local properties of matter (Gross & Perry 1983; Davidson & Owen 1985; Wesson 1992a, b). This means that for stellar systems other than ours, there could well be significant effects of an extra dimension to be found.

It therefore seems worthwhile to obtain data from other astrophysical systems; specifically, to look for nonstandard light deflections in lensed systems such as QSOs, and for anomalous perihelion shifts in eclipsing binaries such as DI Herculis (Guinan & Maloney 1985; Maloney, Guinan, & Boyd 1989), binary pulsars (Haugan 1985; Taylor & Weisberg 1989), and possible pulsars with planetary companions (Wolszczan & Frail 1992; Thorset & Phillips 1992). These ways of testing Kaluza-Klein theory may require some manipulation of our equations, but the latter do contain all of the essential physics relevant to the classical tests as modified by an extra dimension. Important further steps in the search for extra dimensions will be to derive predictions for the Nordtvedt effect and gravitational wave damping in the binary pulsar.

To end, we should perhaps ask what is the *most immediate* way to test Kaluza-Klein theory. In this regard, we note that extant data and data likely to be available in the near future involve the weak-field limit. This implies that our formulae are directly applicable, without ambiguities connected with coordinate effects. For example, our starting metric (1) in the solar system involves a difference between coordinate radius and so-called proper radius which at the distance of Earth from the Sun is of order 1 part in 10^7. This is far less than uncertainties in the data for the classical tests, which are typically of order a few parts in 10^3. Also, our fomula (53) identifies the gravitational mass of the Sun as aM from the frequency shift effect. This agrees with what might be called the Kepler mass, defined from Kepler's third law and the motions of the planets. [To see this, one can take our equations (A2.3) and (A2.4) and form $(\dot{\phi}^2/\dot{t}^2)$ to obtain $(d\phi/dt)^2 = aM/r^3$ to order (M/r) in the weak-field limit.] Thus our equations are well defined, and the real question is which of them are most likely to provide a way of distinguishing between four-dimensional Einstein theory and five-dimensional Kaluza-Klein theory. The answer to this is dependent on progress in observational and experimental astrophysics. In the former area, our formula (18.2) could be immediately tested given the appropriate data from gravitational lenses. In the latter area, our formula (66) could be immediately tested given the results of an Earth-orbit gyroscope experiment.

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APPENDIX

A1. CHRISTOFFEL SYMBOLS

The nonzero Christoffel symbols for the metric of equation (1) are

$$\Gamma^0_{10} = \Gamma^0_{01} = \frac{aM}{r(r-2M)} ,$$

$$\Gamma^1_{00} = \frac{aM}{r^2}\left(\frac{1-2M}{r}\right)^{2a+b-1} , \quad \Gamma^1_{11} = \frac{-(a+b)M}{r(r-2M)} , \quad \Gamma^1_{22} = -(1-a-b)M-(r-2M) ,$$

$$\Gamma^1_{33} = -[(1-a-b)M + r - 2M]\sin^2\theta , \quad \Gamma^1_{55} = -\frac{bM}{r^2}\left(\frac{1-2M}{r}\right)^{a+2b-1} ,$$

$$\Gamma^2_{12} = \Gamma^2_{21} = (1-a-b)\frac{M}{r(r-2M)} + \frac{1}{r} , \quad \Gamma^2_{33} = -\sin\theta\cos\theta ,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = (1-a-b)\frac{M}{r(r-2M)} + \frac{1}{r} , \quad \Gamma^3_{23} = \Gamma^3_{32} = \cot\theta ,$$

$$\Gamma^5_{15} = \Gamma^5_{51} = \frac{bM}{r(r-2M)} .$$

A2. CIRCULAR ORBIT EXPRESSIONS

Here we prove the expressions (57) and (58) for $\dot{t}$ and Ω, respectively. Equation (26) for a timelike circular orbit with no motion in the fifth dimension ($\dot{r} = \ddot{r} = \dot{n} = 0$) reads

$$-aA^{a-1}A_{,r}\dot{t}^2 + (1-a-b)A^{-(a+b)}A_{,r}r^2\dot{\phi}^2 + 2A^{(1-a-b)}r\dot{\phi}^2 = 0 . \tag{A2.1}$$

This we can solve for $\Omega \equiv d\phi/dt$ to obtain

$$\Omega = \left[\frac{(1-a-b)}{a}\left(\frac{1-2M}{r}\right)^{1-2a-b} r^2 + \frac{1}{aM}\left(\frac{1-2M}{r}\right)^{2-2a-b} r^3\right]^{-1/2} . \tag{A2.2}$$

Next, from equation (3) we have

$$\dot{t}^2 = \frac{l^2}{A^{2a}} . \tag{A2.3}$$

This into (A2.1) gives

$$\dot{\phi}^2 = \frac{aMA^{a-1}}{(1-a-b)Mr^2A^{-(a+b)} + A^{1-a-b}r^3}\frac{l^2}{A^{2a}} . \tag{A2.4}$$

From equation (1), under our assumptions, we can write

$$A^a\dot{t}^2 - A^{1-a-b}r^2\dot{\phi}^2 = 1 . \tag{A2.5}$$

When combined with equations (A2.3) and (A2.4) this gives

$$l^2\left[A^{-a} - \frac{aA^{-a}M}{(1-a-b)M + Ar}\right] = 1 ,$$

or equivalently

$$l = \left(\frac{1-2M}{r}\right)^{a/2} \Big/ \left[1 - \frac{aM}{r-(1+a+b)M}\right]^{1/2} \tag{A2.6}$$

Then equation (A2.6) into equation (3) yields

$$t = \left(\frac{1-2M}{r}\right)^{-a/2} \Big/ \left[1 - \frac{aM}{r-(1+a+b)M}\right]^{1/2} \tag{A2.7}$$

A3. PERIHELION SHIFT EXPRESSIONS

The perihelion shift for a nearly circular orbit with $k = 0$ can be found from equation (38.1) as follows:

We need to compute d/h^2, e/h^2 for such an orbit. Inserting $k = 0$ into equations (35) and (36) gives

$$d = 1 + (a-1)(-1+2l^2) + b(l^2-1) , \qquad e = 2(2-a-b)(-1+a+b) + 2(-2+2a+b)(-1+2a+b)l^2 , \tag{A3.1}$$

so

$$\frac{d}{h^2} = (2-a-b)\frac{1}{h^2} + (2a+b-2)\frac{l^2}{h^2} , \qquad \frac{e}{h^2} = 2(2-a-b)(-1+a+b)\frac{1}{h^2} + 2(-2+2a+b)(-1+2a+b)\frac{l^2}{h^2} . \tag{A3.2}$$

We already have equations (A2.4) and (A2.6) for ϕ and the constant of motion l correspondingly. The former expression combined with equation (4) gives

$$h^2 = \frac{aA^{1-a-b}Mr^2}{r-(1+2a+b)M} , \tag{A3.3}$$

and then the above equation together with equation (A2.6) yield

$$\frac{l^2}{h^2} = A^{2a+b}\frac{r + A^{-1}(1-a-b)M}{aMr^2} . \tag{A3.4}$$

Finally, equations (A3.2)–(A3.4) and (38.1) enable us to write

$$\delta\phi = \frac{6\pi M}{r}\left(\frac{2-a-b}{a} + \frac{2a+b-2}{a}\right) + \frac{2\pi M}{ar}$$
$$\times [(2-a-b)(-1+a+b) + (-2+2a+b)(-1+2a+b)] + O\left[\left(\frac{M}{r}\right)^2\right] = \frac{6\pi M}{r}\left(a + \frac{2}{3}b\right) . \tag{A3.5}$$

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