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Flat FRW Models with Variable G and Λ

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We consider Einstein's equations with variable gravitational coupling G and cosmological term Λ. For a power-law time-dependence of G, the cosmological term varies in proportion to the inverse square of the time, provided the equation of state is not that of vacuum. There is then no dimensional constant associated with Λ. For a vacuum equation of state the model is compatible with classical inflation for a wide class of functions $G(t)$ and $\Lambda(t)$. For non-power-law behaviour of $G(t)$, it is possible to have a scale factor that increases exponentially without a vacuum equation of state. For this case the energy density associated with Λ decreases exponentially, while at time zero it is equal with opposite sign to the regular energy density, so there is zero total energy initially.

1. INTRODUCTION

There have been numerous extensions of general relativity in which the gravitational parameter G varies with cosmic time t [1]. This is reasonable, since G couples geometry to matter, and in an evolving universe we might expect $G = G(t)$. None of these theories has been widely accepted. But recently a new one has been discussed which may be more appealing, because it leaves the form of Einstein's equations formally unchanged by virtue of allowing a change in G to be accompanied by one in the cosmological parameter Λ [2–5]. This approach is actually non-covariant, but

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we believe it is worth investigating because it may be the limit of a more viable fully covariant theory, such as 5D gravity [6]. In what follows we will look at some solutions in this theory for the case of perfect fluid, flat Friedman-Robertson-Walker models. Certain solutions for this case have already been found [7]. But we will adopt an approach which is more compact than others, and which not only allows us to confirm earlier solutions but also leads to new ones. We will in addition make some comments aimed at clarifying the relevance of these solutions to important cosmological questions such as the horizon problem, quantization and the size of Λ.

2. GENERAL ANALYSIS AND SOLUTIONS WITH POWER-LAW G

We consider a spatially flat FRW universe

$$ds^2 = dt^2 - R(t)^2[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)] \tag{1}$$

with a perfect fluid energy momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{2}$$

and time dependent G and Λ.

Friedman's equations and the law of energy-momentum conservation ($T^{;\nu}_{\mu\nu} = 0$) have the same form as in the standard case,

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G(t)\rho}{3} + \frac{\Lambda(t)}{3} \tag{3}$$

$$\dot{\rho} + 3(1+w)\frac{\dot{R}}{R}\rho = 0, \tag{4}$$

where dot denotes differentiation with respect to t and in eq. (4) we assumed an equation of state $p = w\rho$, $w =$ constant. An additional equation relating the time changes of G and Λ can be obtained by the Bianchi identities $(R_{\mu\nu} - 1/2\, R g_{\mu\nu})^{;\nu} = 0 = (8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu})^{;\nu}$, which under our assumptions yield

$$\dot{\Lambda} = -8\pi \dot{G}\rho. \tag{5}$$

The system of eqs. (3)–(5) requires one more equation for a unique solution. Before specifying this, however, it is convenient to combine (3),

(4) and (5) to give a single relation that applies to all models. (We believe that this is a better way to approach the subject than others in the literature.) Thus combining eqs. (3) and (4) and squaring we obtain

$$\left(\frac{1}{\rho}\dot{\rho}\right)^2 = 9(1+w)^2\left(\frac{8\pi G}{3}\rho + \frac{\Lambda}{3}\right), \tag{6}$$

and by differentiating with respect to the time t and using eqs. (4) and (5) we get

$$\frac{\dot{\rho}\ddot{\rho}}{\rho^2} - \left(\frac{\dot{\rho}}{\rho}\right)^3 = 12(1+w)^2\pi G\dot{\rho}. \tag{7}$$

This is the central equation of our analysis. For the moment we assume that the density is not a constant. (We will return to this below.) We can then write

$$\rho\ddot{\rho} - \dot{\rho}^2 = 12(1+w)^2\pi G\rho^3. \tag{8}$$

Let us now assume that $G(t)$ is given by a power law

$$G = Ct^n, \qquad C = \text{const.} \tag{9}$$

We can now solve the system. From (8) and (9) we have

$$\rho(t) = \frac{n+2}{12(1+w)^2\pi C}\frac{1}{t^{n+2}}, \qquad n \neq -2. \tag{10}$$

For the density to be positive definite we must have that $n > -2$. Next, on combining eqs. (5), (8) and (9) we obtain

$$\dot{\Lambda} = -\frac{2n(n+2)}{3(1+w)^2}\frac{1}{t^3}, \tag{11}$$

which on integrating and setting a constant to zero (for self consistency of the system) gives

$$\Lambda = \frac{n(n+2)}{3(1+w)^2}\frac{1}{t^2} \qquad n \neq -2. \tag{12}$$

We see that Λ varies as t^{-2}, which matches its natural dimensions and means there is no longer a dimensional constant associated with the cosmological term in the field equations. [Note that this result follows from eq. (7), but if we were to treat the case of a finite constant Λ then we have to return to eq. (6).] Following from (12), this with (9) and (10) into (3) gives

$$R(t) = \text{const.} \times t^{(n+2)/[3(1+w)]}, \qquad n \neq -2. \tag{13}$$

We can see that for $n = 0$ this reduces to our familiar result $R \propto t^{2/[3(1+w)]}$ for the flat FRW model with constant G.

3. SOLUTIONS WITH CONSTANT DENSITY

Obviously, one solution of eq. (7) is $\dot{\rho} = 0$. One way this can be compatible with eq. (4) is if the scale factor is a constant too. Equation (3) then yields

$$8\pi G\rho = -\Lambda, \tag{14}$$

whose time derivative is (5), showing that our system of equations is satisfied. (The functions $G(t)$ and $\Lambda(t)$ in eq. (14) are arbitrary.) The vacuum energy in this solution is

$$\rho_{\text{vac.}} = \frac{\Lambda}{8\pi G} = -\rho \tag{15}$$

and if we denote the total energy by $\rho_{\text{tot.}} \equiv \rho + \rho_{\text{vac.}}$ we have

$$\rho_{\text{tot.}} = 0. \tag{16}$$

Therefore, we have a spatially flat static universe with varying G and Λ, and zero total energy. One can prove that if $n = -2$ and $w \neq -1$ the above solution is necessary. For that value of n, however, we can see from eq. (14) that Λ still falls as $1/t^2$ and, therefore, this law holds for every $n \geq -2$.

An alternative way of satisfying eq. (4) with a constant energy density is to have $w = -1$ so R can be nonzero. Using $\dot{\rho} = 0$ and eq. (5), the time derivative of eq. (3) yields

$$R\ddot{R} - \dot{R}^2 = 0. \tag{17}$$

This equation has solutions of the form $R = \exp(\pm\text{const.} \times t)$ implying classical inflation (De Sitter expansion). Note that the only constraint on G and Λ comes from eq. (5) and as in the case of the static solutions their functional form is otherwise free. This means that for any $G(t)$ and $\Lambda(t)$ satisfying (5), we obtain inflationary solutions from the assumption of a vacuum equation of state. It should be noted that Berman [7], in a treatment of the same subject we are considering but by a somewhat different method, has emphasized the avoidance of the horizon problem with solutions where the scale factor is proportional to the time.[4] However, it

[4] Incidentally, our analysis for power-law $G(t)$ leads to the same equations as Berman's though with a different parametrization, but we note that in his eqs. (16)–(18) α should be

$$\alpha = \left[\frac{m}{3}\left(2 + \frac{B}{4\pi A}\right)\right] - 1$$

for consistency.

is apparent from our eq. (17) that the horizon problem in universes with variable G and Λ can also be avoided with solutions of the traditional exponential kind. Furthermore, while a variable Λ-term of geometric origin, as considered in these models, can provide a more flexible framework for the solution of the cosmological constant problem, it is the behavior of the sum of Λ with the contribution to it from the rich vacuum of gauge theories that has to be consistent with particle physics predictions and observation [8]. Further study is needed before one can claim successful resolution of the problem within this framework.

4. OTHER SOLUTIONS

Solutions to the differential equation for the energy density, eq. (8), can also be found for forms $G(t)$ other than power law. In particular, consider

$$G(t) = Bt^{s-2}\exp(-bt^{s}), \qquad B > 0, \tag{18}$$

and try for the energy density

$$\rho(t) = A\exp(qt^{s}). \tag{19}$$

On inserting the above expressions into eq. (8) we obtain

$$A^2 s(s-1)qt^{s-2}\exp(2qt^{s}) = 12(1+w)^2\pi BA^3t^{s-2}\exp[(3q-b)t^{s}], \tag{20}$$

which can be solved for A and q to give

$$\rho(t) = \frac{s(s-1)b}{12(1+w)^2\pi B}\exp(bt^{s}). \tag{21}$$

For the energy density to be positive we must have

$$bs(s-1) > 0. \tag{22}$$

The cosmological constant can now be obtained from eq. (5), which gives,

$$\Lambda = -\frac{2s(s-1)b}{3(1+w)^2}\left[t^{s-2} - \frac{bs}{2(s-1)}t^{2(s-1)}\right], \tag{23}$$

with the scale factor

$$R(t) = \text{const.} \times \exp\left[-\frac{bt^{s}}{3(1+w)}\right]. \tag{24}$$

In order to have expansion b has to be negative. Note that these are exponential solutions that do not arise from an equation of state $p = -\rho$ as in inflation, because $w = -1$ is not allowed in the above expressions. Another interesting feature is that the vacuum energy density is exponentially supressed when we have expansion ($b < 0$), since from eqs. (18) and (23) we have

$$\rho_{\rm vac} = -\frac{s(s-1)b}{12\pi B(1+w)^2}\left[1 - \frac{bs}{2(s-1)}t^s\right]\exp(bt^s). \tag{25}$$

The ratio of the ordinary energy density to the vacuum one is

$$\frac{\rho}{\rho_{\rm vac}} = -\frac{1}{1 - [bs/2(s-1)]t^s}, \tag{26}$$

so in the limit of $t \to 0$ we have $\rho = -\rho_{\rm vac}$. This corresponds to an initial state with zero total energy. The scale factor and the energy densities ρ and $\rho_{\rm vac}$ are finite in this limit, but both Λ and G diverge at time zero for the expanding solutions. It should be noted that solutions like ours with G divergent for $t \to 0$, (except the power law ones), provide a classical rational for strong-coupling quantum gravity, in which the dynamical part of the Hamiltonian is effectively reduced from two terms to one by letting G diverge [9,10]. Of course, for models in which G was different in the past it may be necessary to restrict them to epochs prior to nucleosynthesis, and in general to choose the parameters that are compatible with limits on the variation of G [11].

5. CONCLUSION

We have looked at Einstein's equations for the case of spatially flat FRW models where the perfect-fluid energy-momentum tensor has zero divergence, but where the gravitational parameter G and cosmological parameter Λ are both allowed to depend on the cosmic time t. We have confirmed some solutions in the literature and found some new ones with interesting properties. We have discussed possible connections with well known questions in gravitation, notably the size of the cosmological constant and quantum gravity and have shown the compatibility of the model with classical inflation. However, we have concentrated on the solutions, and their detailed physical implications will require further investigation.

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