Analytical approach to spherically-symmetric solutions of the Einstein scalar field equations II. Perturbations

J. E. Berberian[*] and A. S. Silbergleit
Gravity Probe B
W. W. Hansen Experimental Physics Laboratory
Stanford University, Stanford, CA 94305-4085

We continue our study of the Einstein scalar field equations by investigating asymptotic expansions to the spherically symmetric solutions, where we use the coupling of the scalar field to the geometry as our small parameter. Examples of this kind of asymptotic expansion are given using known exact solutions to the system. Although there are exact solutions that do not have asymptotic expansions of this type, those that do can be classified into two distinct categories: perturbations about Schwarzschild, and perturbations about the degenerate metric $ds^2 = r^2 d\Omega^2$. Each category is examined in both scalar field and Schwarzschild coordinates. The Schwarzschild category gives the behaviour expected from previous studies, but the degenerate category is altogether new, and exhibits properties reminiscent of Mach's principle.

1 Introduction

In [1] we introduced scalar field coordinates to study the problem of spherically-symmetric Einstein equations:

$$R_{\mu\nu} = 2\ \phi_\mu\ \phi_\nu$$
$$\Box_g \phi = 0 \tag{1}$$

where the metric and scalar field are given by:

$$ds^2 = g_{tt}(r,t)dt^2 + g_{rr}(r,t)dr^2 + r^2 d\Omega^2$$
$$\phi = \phi(r,t) \tag{2}$$

The process of constructing the coordinates consisted of two steps: rescaling the scalar field:

$$\phi = \frac{\varepsilon}{\sqrt{2}}\varphi \tag{3}$$

and then choosing coordinates:

$$p = \varphi(r,t),\ q = \psi(r,t) \tag{4}$$

such that the resulting expression for the metric in the new coordinates was diagonal. The line element then took the form:

$$ds^2 = -\frac{1}{2}De^{-b(p,q)}dq^2 + \frac{1}{2}De^{b(p,q)}dp^2 + e^{b(p,q)}d\Omega^2 \tag{5}$$

where

$$D \equiv b_{,pp} - \frac{1}{2}\left(e^{2b}\right)_{,qq} \tag{6}$$

The inverse coordinate transformation was given by:

$$r = e^{\frac{b}{2}}, \; t = h(p,q) \tag{7}$$

$$h_{,p}\, b_{,p} - e^{2b} h_{,q}\, b_{,q} = 0 \tag{8}$$

The field equations in the scalar field coordinates became:

$$R_{\mu\nu} = \varepsilon^2 \delta_{\mu p} \delta_{\nu p} \tag{9}$$

The difference between (9) and the equations $R_{\mu\nu} = 0$ for vacuum is that former contains one equation whose right hand side is not zero but rather ε^2. It is tempting therefore to examine what happens as we make ε small. One would expect that many solutions of (9) possess an asymptotic expansion in ε^2 about the vacuum solution. In this paper we now wish to examine the limit $\varepsilon \to 0$. To do this, we introduce the concept of a *parameter* limit.

2 Parameter limits

Many of the solutions currently known to (1) belong to families whose members are identified by a parameter k. If the vacuum metric belongs to such a family, and therefore is associated with some parameter value k_0, then we introduce ε as a function of the parameter k such that $\varepsilon(k_0) = 0$. As we let ε approach zero, we pass through different solutions in a family to arrive at the vacuum metric. In this way we avoid any complications associated with blindly letting ε equal zero. The limit is restricted to solutions where it makes sense, in order that the properties of these solutions become clear when ε becomes small.

3 Examples of parameter limits

We now give a two examples of how parameter limits function in known families of solutions to (1).

3.1 Static Solution

When $k = 1$, this solution gives us the Schwarzschild geometry. The natural way to introduce ε is to set

$$\varepsilon(k) = \frac{\sqrt{1-k^2}}{\sqrt{2}} \tag{11}$$

Then, our scalar field coordinate p becomes:

$$p = \frac{\sqrt{2}}{\varepsilon}\phi = \log\left(1 - \frac{2M}{\rho}\right) \tag{12}$$

which is still a valid coordinate for $\varepsilon = 0$, as desired.

3.2 A Non-Static Solution

As a second example, let us consider the solution [3]:

$$\begin{aligned} ds^2 &= -d\tau^2 + d\rho^2 + \left(1 - \tfrac{kt}{\rho}\right) d\Omega^2 \\ \phi &= \tfrac{1}{2}\log\left(1 - \tfrac{kt}{\rho}\right) \end{aligned} \tag{13}$$

For this family, the vacuum solution appears when $k = 0$. Therefore, we set:

$$k(\varepsilon) = \varepsilon \tag{14}$$

Then our coordinate p becomes:

$$p = \frac{1}{\sqrt{2}\varepsilon}\log\left(1 - \frac{\varepsilon t}{\rho}\right) \tag{15}$$

As $\varepsilon \to 0$, $p \to -\frac{t}{\sqrt{2}\rho}$, which again is a valid coordinate.

3.3 Parameter-Free Solution

Not all solutions to (1) have a vacuum limit, which means that some solutions cannot be considered to be perturbations of the vacuum solution. For completeness, we give an example of such a solution [2]:

$$\begin{aligned} ds^2 &= -2r^2dt^2 + 2dr^2 + r^2d\Omega^2 \\ \phi &= t \end{aligned} \tag{16}$$

Since this solution has no free parameters, we cannot make it arbitrarily close to a vacuum solution, which means that we cannot approximate it with a perturbation expansion about a vacuum solution.

4 Classification of the perturbations

We now turn to the field equations. The governing equations can be expressed in the form:

$$z_{,p} = b_{,p}\, z\left[\frac{\varepsilon^2}{\Delta} - \left(z - \frac{1}{2}\right)\right] \tag{17}$$

$$z_{,q} = -b_{,q}\, z\left[\frac{\varepsilon^2}{\Delta} + \left(z - \frac{1}{2}\right)\right] \tag{18}$$

where D is given in (6), and

$$\Delta \equiv (b_{,p})^2 - e^{2b}(b_{,q})^2, \; z \equiv D/\Delta \tag{19}$$

The unknown function $z(p,q)$ is related to the variable Schwarzschild mass $\mu(p,q)$ introduced in [1], equation(12), by

$$z = \frac{1}{2}\left(1 - \frac{\mu}{r}\right)^{-1}. \tag{20}$$

We expand $b(p,q)$ and $z(p,q)$ in terms of a power series in ε^2:

$$\begin{aligned} b &= b^{(0)} + \varepsilon^2 b^{(1)} + \cdots \\ z &= z^{(0)} + \varepsilon^2 z^{(1)} + \cdots \end{aligned} \tag{21}$$

and thus obtain equations for $z^{(0)}$:

$$\begin{aligned} z^{(0)}{}_{,p} &= -b^{(0)}{}_{,p}\, z^{(0)}\left(z^{(0)} - \tfrac{1}{2}\right) \\ z^{(0)}{}_{,q} &= -b^{(0)}{}_{,q}\, z^{(0)}\left(z^{(0)} - \tfrac{1}{2}\right) \end{aligned} \tag{22}$$

We can solve these equations exactly, and there are three cases: $z^{(0)} \equiv 0$, $z^{(0)} \equiv \frac{1}{2}$, and the general case when neither of these is true.

4.1 General Case

If $z^{(0)} \not\equiv 0, \frac{1}{2}$, then

$$\left[2\log\frac{z^{(0)}}{z^{(0)} - 1/2} - b^{(0)}\right]_{,p} = \left[2\log\frac{z^{(0)}}{z^{(0)} - 1/2} - b^{(0)}\right]_{,q} = 0 \tag{23}$$

or,

$$2z^{(0)} = \frac{e^{\frac{b}{2}}}{e^{\frac{b}{2}} - 2M^{(0)}} \tag{24}$$

where $M^{(0)}$ is a constant. This is the Schwarzschild solution, so this is the class of perturbations about the Schwarzschild geometry.

4.2 Case $z^{(0)} = \frac{1}{2}$

If $z^{(0)} = \frac{1}{2}$, then $2z^{(0)} = 1$, which corresponds to equation (24) with $M^{(0)} = 0$. So, this class of perturbations are expansions about the Minkowski geometry.

4.3 Case $z^{(0)} = 0$

This case implies that $D^{(0)} = \Delta^{(0)} z^{(0)} = 0$. The line element to zeroth order is:

$$\left(ds^{(0)}\right)^2 = r^2 d\Omega^2 \tag{25}$$

which is the degenerate, two-dimensional metric. Therefore, the metric coefficients that are missing in (25) are due entirely to the coupled scalar field; in this case, one can say that the scalar field creates the four-dimensional spacetime.

Thus we may categorize the solutions of our system in the following way: a) solutions with *no* parameter limit, b) solutions which are asymptotic expansions about the Schwarzschild geometry, and c) solutions which are asymptotic expansions about the degenerate metric $ds^2 = r^2 d\Omega^2$. It is a nice feature of scalar field coordinates that the asymptotic expansions can be classified so cleanly.

5 Perturbations about the Schwarzschild geometry

Since we have solved for the $z^{(0)}$, we now turn to the problem of solving for $b^{(0)}$:

$$D^{(0)} = z^{(0)}\Delta^{(0)} \tag{26}$$

Substituting for $z^{(0)}$ from (24), we obtain

$$b^{(0)}{}_{,pp} - \frac{1}{2}\left(e^{2b^{(0)}}\right)_{,qq} = \frac{\exp\left(b^{(0)}/2\right)\left[\left(b^{(0)}{}_{,p}\right)^2 - e^{2b^{(0)}}\left(b^{(0)}{}_{,q}\right)^2\right]}{2\left[\exp\left(b^{(0)}/2\right) - 2M^{(0)}\right]} \tag{27}$$

Unfortunately, this is a non-linear, second order equation, even in the special case when $M^{(0)} = 0$. Since we cannot find its general solution, we have little hope of solving explicitly for $b^{(1)}$, although several particular solutions have been found. However, the first correction $\mu^{(1)}$ to the variable Schwarzschild mass can be immediately expressed in terms of integrals of $b^{(0)}$. Then, by equation (14) from [1], the formula for the line element in Schwarzschild coordinates is readily available. The same is done in a slightly easier way in {r,t} coordinates, and the proper expressions for the metric coefficients to the same order in ε are:

$$\begin{aligned} g_{rr}(r,t) = \left(1 - \frac{2M^{(0)}}{r}\right)^{-1} \Bigg[1 + \varepsilon^2 \Bigg\{ L(r,t_0) + 2rK(r,t) + \frac{C_1}{r - 2M^{(0)}} \\ - \frac{1}{r - 2M^{(0)}} \int_{r_0}^{r} L(r',t_0)dr' \Bigg\}\Bigg] \\ g_{tt}(r,t) = -\left(1 - \frac{2M^{(0)}}{r}\right) \Bigg[1 + \varepsilon^2 \Bigg\{ 2rK(r,t) + L(r,t_0) + 2g(t) + \frac{C_2}{r - 2M^{(0)}} \\ + \frac{1}{r - 2M^{(0)}} \int_{r_0}^{r} L(r',t_0)dr' + 4\int_{r_0}^{r} \frac{r'}{r' - 2M^{(0)}} K(r',t)dr' \Bigg\}\Bigg] \end{aligned} \tag{28}$$

where

$$\begin{aligned} L(r,t) &\equiv \int_{r_0}^{r} \left[x\left(\phi_{,r}^{(1)}(x,t)\right)^2 + \frac{x^3}{\left(x - 2M^{(0)}\right)^2}\left(\phi_{,t}^{(1)}(x,t)\right)^2 \right] dx \\ K(r,t) &\equiv \int_{t_0}^{t} \left[\phi_{,r}^{(1)}(r,y)\,\phi_{,t}^{(1)}(r,y)\right] dy, \end{aligned} \tag{29}$$

$g(t)$ is an arbitrary function of time, and C_1 and C_2 are arbitrary constants. The first term $\phi^{(1)}$ of the scalar field expansion,

$$\phi = \varepsilon\phi^{(1)} + \varepsilon^3\phi^{(2)} + \cdots \tag{30}$$

satisfies the wave equation with the unperturbed Schwarzschild metric:

$$\phi_{,tt}^{(1)} = \frac{r - 2M^{(0)}}{r^3} \left[\left(r - 2M^{(0)} \right) r\phi_{,r}^{(1)} \right]_{,r} \tag{31}$$

Even though this equation — unlike (27) — is linear, its general explicit solution is not known. So, for the general case, we are forced to express the metric corrections in terms of integrals of functions which we do not know explicitly. In this respect, Schwarzschild coordinates are no better than scalar field coordinates.

5.1 Perturbations about the Minkowski geometry

We simplify the situation by setting $M^{(0)} = 0$. Then (31) is just the normal flat space wave equation, whose general solution is:

$$\phi^{(1)} = \frac{g(r+t) + h(r-t)}{r} \tag{32}$$

where $g(x)$ and $h(x)$ are arbitrary smooth enough functions. Next, we let $g(x)$ and $h(x)$ be pulses, namely:

$$\int_{-\infty}^{\infty} [g'(x)]^2 \, dx < +\infty, \int_{-\infty}^{\infty} [h'(x)]^2 \, dx < +\infty \tag{33}$$

and $g(\pm\infty) = 0$, $h(\mp\infty) = 0$. We take $t_0 = -\infty$, $C_1 = C_2 = 0$, and $g'(-\infty) = h'(+\infty) = 0$ in (28) and (29) so that $g_{\mu\nu}(r, t_0) = \eta_{\mu\nu}$. Then the solution takes the form:

$$\begin{aligned} g_{tt} &= -\left[1 + \varepsilon^2 \left\{ \frac{2m(r,t)}{r} + n(r,t) \right\}\right] \\ g_{rr} &= \left[1 + \varepsilon^2 \left\{ \frac{2m(r,t)}{r} - \left[\phi^{(1)}(r,t)\right]^2 \right\}\right] \end{aligned} \tag{34}$$

where

$$\begin{aligned} m(r,t) &\equiv \int_{-\infty}^{r+t} [g'(x)]^2 \, dx - \int_{r-t}^{\infty} [h'(x)]^2 \, dx \\ M^{(1)} &= \lim_{t\to\infty} m(r,t) \\ n(r,t) &= 4\int_{r_0}^{r} \frac{1}{x^2} \left[m(x,t) - x \left\{ \phi^{(1)}(x,t) \right\}^2 \right] dx - \frac{4M^{(1)}}{r_0} \end{aligned} \tag{35}$$

If, in addition, $h(x)$ and $g(x)$ satisfy $g'(+\infty) = h'(-\infty) = 0$, then as $t \to \infty$, $n(r,t) \to -4M^{(1)}/r$, yielding the line element:

$$ds^2 = -\left(1 - \varepsilon^2 \frac{2M^{(1)}}{r}\right) dt^2 + \left(1 + \varepsilon^2 \frac{2M^{(1)}}{r}\right) dr^2 + r^2 d\Omega^2 \tag{36}$$

In this example we clearly see that the first order perturbation yields the answer we expected: the difference in the energies of the incoming and outgoing scalar fields is left behind as a Schwarzschild mass, and the metric evolves with time to the Schwarzschild solution. Qualitatively, this is what one would expect, since Christodoulou has shown under certain conditions that pure scalar field collapse in asymptotically flat space-times results in a Schwarzschild or Minkowski geometry [4].

6 Perturbations about the degenerate solution $ds^2 = r^2 d\Omega^2$

In this case, since $D^{(0)} = 0$, we have $z^{(0)} = 0$. Therefore, the zero order equations are automatically satisfied, and we proceed to the next order:

$$z,_p^{(1)} = \frac{1}{2} b,_p^{(0)} z^{(1)} \tag{37}$$

$$z,_q^{(1)} = \frac{1}{2} b,_q^{(0)} z^{(1)} \tag{38}$$

We can immediately integrate these equations to yield:

$$z^{(1)} = \beta e^{\frac{b^{(0)}}{2}} = \beta r, \; \beta = const \tag{39}$$

Next we examine the equation for $b^{(0)}$, which we will need in order to solve (39) for $b^{(1)}$. For $b^{(0)}$ we have:

$$D^{(0)} = b,_{pp}^{(0)} - \frac{1}{2} \left(e^{2b^{(0)}} \right),_{qq} = 0 \tag{40}$$

with the additional constraint that $\Delta^{(0)} \not\equiv 0$. Again, the general solution to this equation is unknown, but this does not prevent us from expressing the line element in Schwarzschild coordinates to first order using our {p,q} formalism. Namely, from (20) and (21), we immediately find the variable Schwarzschild mass:

$$\mu = r \left(1 - \frac{1}{2\varepsilon^2 z^{(1)} + \cdots} \right) = r \left(1 - \frac{1}{2\varepsilon^2 \beta r + \cdots} \right) \tag{41}$$

and finally, by equation (14) from [1] the line element in Schwarzschild coordinates becomes:

$$ds^2 = \varepsilon^2 C \left\{ -\frac{1}{2r} dt^2 + 2r dr^2 \right\} + r^2 d\Omega^2, \; C = const \tag{42}$$

It is interesting to note that as $\varepsilon \to 0$, μ diverges. The scalar field $\phi^{(1)}$ satisfies the equation

$$r^3 \phi,_{tt}^{(1)} = \phi,_r^{(1)} + r \phi,_{rr}^{(1)}, \tag{43}$$

which allows for a general solution in terms of Bessel functions.

The interesting thing about this class of solutions is that they undergo a dimensional reduction when the scalar field coupling vanishes. That is, the scalar field generates the extra two dimensions. This suggests an example of a kind of Mach's principle in which the presence of matter (e.g. scalar field) is crucial to the existence of a four-dimensional space-time.

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[*] Also at Department of Physics, Stanford University.

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