Analytical approach to spherically symmetric solutions of the Einstein scalar field equations I.

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We investigate the Einstein scalar-field equations $R_{\mu\nu} = 2\phi_{,\mu}\,\phi_{,\nu}$ in spherical symmetry using the scalar field itself as a coordinate. We construct the coordinate system by requiring only that $\Box_g \phi = 0$, and ϕ is not constant. The metric coefficients in the new coordinates are shown to be related to the Ricci scalar. The field equations simplify to yield a compatible system of two first-order partial differential equations for two functions $\mu(p,q)$ and $b(p,q)$, where $\mu(p,q)$ is the position-dependent Schwarzschild mass of the system, and $b(p,q)$ is the logarithm of the areal metric coefficient. Exact solutions are given for special cases: we recover the Wyman solutions by requiring that $b_{,p} = 0$ or $b_{,q} = 0$.

1 Introduction

The problem of a massless scalar field coupled to gravity, described by the system of equations $R_{\mu\nu} = 2\ \phi_{,\mu}\ \phi_{,\nu}$ has been well-studied in spherical symmetry, and with good reason. In general relativity, this system represents a very simple matter model which one can use to investigate gravitational collapse and black hole formation, where recent numerical studies have produced indications of scaling phenomena [1]. In scalar-tensor theories of gravity, this system represents (after a conformal transformation) an even simpler matter model: spherically symmetric vacuum. Because of the recent activity in scalar-tensor theories [2], it is important to understand the simplest example of a scalar-tensor space-time.

This system has been explored in many different coordinate systems: Bondi [3], Schwarzschild [4], and double-null [5]. In this paper, we open a new avenue of investigation by introducing the scalar field itself as a coordinate.

2 Scalar Field Coordinates

Because of our restriction to spherical symmetry, the metric and scalar field can be expressed in the form:

$$ds^2 \;=\; g_{tt}(r,t)\ dt^2 \;+\; g_{rr}(r,t)\ dr^2 \;+\; g_{\theta\theta}(r,t)\ d\Omega^2$$
$$\phi = \phi(r,t). \tag{1}$$

We simplify our equations by choosing new coordinates in two steps. First, we rescale the scalar field: $\phi = \frac{\varepsilon}{\sqrt{2}}\,\varphi$. Under this rescaling, the field equations become:

$$R_{\mu\nu} = \varepsilon^2 \, \varphi_{,\mu} \, \varphi_{,\nu} \tag{2}$$

Second, assuming that $\varphi(r,t)$ is not a constant, we choose new coordinates $p = \varphi(r,t)$ and $q = \psi(r,t)$ such that the resulting metric is diagonal. Using the fact that $\Box_g \varphi = 0$, it can be shown that $\psi(r,t)$ must obey the following compatible first order system:

$$\begin{aligned} \psi_{,r} &= -g_{\theta\theta}\sqrt{-g_{tt}g_{rr}}\; g^{tt}\varphi_{,t} \\ \psi_{,t} &= g_{\theta\theta}\sqrt{-g_{tt}g_{rr}}\; g^{rr}\varphi_{,r} \end{aligned} \tag{3}$$

Expressed in the new coordinates, the metric then has the form:

$$ds^2 \;=\; -C(p,q)\; dq^2 \;+\; A(p,q)\; dp^2 + \; B(p,q)\; d\Omega^2, \tag{4}$$

where $r^2 = B(\varphi(r,t), \psi(r,t))$.

The clear advantage to using these coordinates is the fact that the scalar field derivatives become $\phi_{,p} = \varepsilon/\sqrt{2}$, $\phi_{;q} = 0$, thus eliminating one of the unknown functions of the problem. The field equations (2) then reduce to:

$$R_{\mu\nu} = \varepsilon^2 \; \delta_{\mu p} \; \delta_{\nu p} \tag{5}$$

Rescaling is important because it allows us to study the limit $\varepsilon \to 0$, where we expect to regain the solutions to the vacuum equations. This we shall do in Part II.

It is useful at this point to consider a specific example. A well-known family of static solutions to our system are the metrics:

$$\begin{aligned} ds^2 \; &= -(1 - \tfrac{2M}{\rho})^k \, dt^2 \; + (1 - \tfrac{2M}{\rho})^{-k} \, d\rho^2 \; + (1 - \tfrac{2M}{\rho})^{1-k} \; \rho^2 \; d\Omega^2 \\ \phi(\rho, t) \; &= \; \tfrac{\sqrt{1-k^2}}{2} \; \log\left(1 - \tfrac{2M}{\rho}\right) \\ & 0 \le k \le 1, \;\; \rho > 2M \end{aligned} \tag{6}$$

We set

$$\begin{aligned} p &= \tfrac{\sqrt{2}}{\varepsilon}\phi = \tfrac{1}{\lambda} \log\left(1 - \tfrac{2M}{\rho}\right) \\ q &= \tfrac{2M}{\lambda} t \\ \lambda &\equiv \varepsilon \; \sqrt{\tfrac{2}{1-k^2}} \end{aligned} \tag{7}$$

In $\{p, q\}$ coordinates, the metric becomes:

$$ds^2 = -\frac{\lambda^2}{4M^2} \exp\left(k\lambda p\right) dq^2 + \frac{4M^2\lambda^2 \exp\left[(2-k)\,\lambda p\right]}{\left(1 - \exp \lambda p\right)^4} dp^2 + \frac{4M^2 \exp\left[(1-k)\,\lambda p\right]}{\left(1 - \exp \lambda p\right)^2} d\Omega^2 \tag{8}$$

3 Field Equations

We now investigate the field equations (5) in scalar field coordinates. The wave equation $\Box_g \phi = 0$ gives us the *algebraic* relation $C = A/B^2$. The equation $R_{\theta\theta} = 0$ provides an immediate expression for A in terms of the metric function B; if we define $b(p,q) \equiv \log B(p,q)$, and $D \equiv b,_{pp} - \frac{1}{2}\left(e^{2b}\right),_{qq}$ then we have:

$$A(p,q) = \frac{1}{2} D e^{b}. \tag{9}$$

Thus, all of the metric functions depend only on $B(p,q)$ and its derivatives. Incorporating these initial results into the line element gives us:

$$ds^2 = -\frac{1}{2} D e^{-b} dq^2 + \frac{1}{2} D e^{b} dp^2 + e^{b} d\Omega^2. \tag{10}$$

The remaining three field equations can be shown to be valid if the following two equations hold:

$$\begin{array}{c} b,_p D,_p + e^{2b} b,_q D,_q = D\left[\varepsilon^2 + \left(b,_{pp} + \frac{1}{2} b,_p^2\right) + e^{2b}\left(b,_{qq} + \frac{1}{2} b,_q^2\right)\right] \\ b,_q D,_q + b,_p D,_q = D\left(2b_{pq} + b,_p b,_q\right) \end{array} \tag{11}$$

We simplify the form of (11) by introducing an unknown function $\mu(p,q)$:

$$\begin{aligned} D &= \frac{e^{\frac{b}{2}}}{2\left(e^{\frac{b}{2}} - \mu\right)} \Delta \\ \Delta &\equiv b_p^2 - e^{2b} b_q^2 \end{aligned} \tag{12}$$

The governing equations then become:

$$\begin{aligned} \mu,_p &= \varepsilon^2 \frac{b,_p}{\Delta}\left(e^{\frac{b}{2}} - \mu\right) \\ \mu,_q &= -\varepsilon^2 \frac{b,_q}{\Delta}\left(e^{\frac{b}{2}} - \mu\right) \end{aligned} \tag{13}$$

We require that $\Delta \not\equiv 0$ in order to ensure non-degenerate solutions. It is easy to show that $\Delta \equiv 0 \Rightarrow D \equiv 0$, which by inspection of (10) yields [6] the unphysical line element $ds^2 = r^2 d\Omega^2$.

We can gain a better understanding of what $\mu(p,q)$ means by expressing the metric coefficients of the $\{r,t\}$ coordinate system in terms of μ and $r = e^b$:

$$ds^2 = -\frac{\exp(2k)}{r(r-\mu)} dt^2 + \left(1 - \frac{\mu}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{14}$$

where

$$\begin{aligned} k &\equiv \int^{r} \frac{d\rho}{\rho - \mu}\bigg|_{t=const} \\ \mu &\equiv \mu(p(r,t), q(r,t)) \end{aligned} \tag{15}$$

Clearly, μ is twice the mass contained in the coordinate radius r at time t. Should $\mu = const$, then it is clear that the metric above reduces to the Schwarzschild metric. Examining our governing equations (13), we see that $\mu(p,q) = const$ is indeed a solution only when $\varepsilon = 0$, just as one would expect.

The most important property of our governing equations is that their compatibility condition

$$\mu_{,pq} - \mu_{,qp} = 0 \tag{16}$$

yields nothing more than our definition of μ given in (12). Consequently, we can view our governing equations as two first-order partial differential equations for two *independent* functions, whose compatibility condition ensures that the two functions are related in the correct way. This allows us to drop the definition of μ and simply seek solutions to (13), since any solution to it must automatically satisfy the compatibility condition (16).

4 Special Solutions

If we demand that $b(p,q)$ depend only on p, or only on q, then we obtain the static solutions found by Wyman [7]. When $b_{,p} \equiv 0$, the field equations are easily integrated to yield (8), which are the solutions when both the metric and ϕ are time-independent. The other case, $b_{,q} \equiv 0$, gives us the static solutions with ϕ proportional to t. This class of solutions has two members, a parametrized family (see [7]), and the lone solution:

$$\begin{aligned} ds^2 &= -2r^2 dt^2 + 2dr^2 + r^2 d\Omega^2 \\ \phi &= t \end{aligned} \tag{17}$$

This last solution is very important, for it has no free parameters, and therefore does not have a vacuum limit. We will deal more thoroughly with this example in Part II.

5 The Meaning of D and Δ

It turns out that D and Δ have physical and mathematical meaning. D can be expressed in terms of the Ricci curvature scalar:

$$R = g^{\alpha\beta} R_{\alpha\beta} = \varepsilon^2 g^{pp} = \frac{2\varepsilon^2}{rD} \tag{18}$$

and Δ can be expressed in terms of the jacobian of the transformation from Schwarzschild to scalar field coordinates:

$$J = \frac{\partial p}{\partial r}\frac{\partial q}{\partial t} - \frac{\partial p}{\partial t}\frac{\partial q}{\partial r} = -\frac{2b_{,p}}{rh_{,q}\,\Delta} \tag{19}$$

where $t = h(p,q)$, $r = \exp\left(b(p,q)/2\right)$ is the transformation from scalar field coordinates to Schwarzschild coordinates.

6 Derivation of a Master Equation for $b(p, q)$

As one might expect from previous treatments of the problem [3], it is possible to derive a single master equation for $b(p,q)$. To do this, we proceed by integrating both equations for μ separately, and then equating the resulting expressions for μ. This is possible because our governing equations are *linear* in μ.

The resulting equation is:

$$\begin{aligned} &\int_{p_0}^{p} b_{,p}(p',q) \exp\left\{ \frac{b(p',q)}{2} + \varepsilon^2 \left[u(p',q) - u(p,q) \right] \right\} dp' - \\ &\quad \int_{q_0}^{q} b_{,q}(p,q') \exp\left\{ \frac{b(p,q')}{2} - \varepsilon^2 \left[v(p,q') - v(p,q) \right] \right\} dq' \\ &= 2\left[Q(q) - \tilde{Q}(q) \right] e^{-\varepsilon^2 u(p,q)} - 2\left[P(p) - \tilde{P}(p) \right] e^{\varepsilon^2 v(p,q)} \end{aligned} \tag{20}$$

where

$$\begin{aligned} u(p,q) &= \int_{p_0}^{p} \frac{b_{,p}(x,q)}{\Delta\left(b(x,q), b_{,p}(x,q), b_{,q}(x,q)\right)} dx \\ v(p,q) &= \int_{q_0}^{q} \frac{b_{,q}(p,y)}{\Delta\left(b(p,y), b_{,p}(p,y), b_{,q}(p,y)\right)} dy \\ Q(q) &= \mu(p_0,q), \;\; P(p) = \mu(p,q_0) \\ \tilde{Q}(q) &= \exp\left[\frac{b(p_0,q)}{2}\right], \;\; \tilde{P}(p) = \exp\left[\frac{b(p,q_0)}{2}\right] \end{aligned} \tag{21}$$

7 Conclusion

We have presented a novel approach to the Einstein scalar field equations using a completely new coordinate system. As with other approaches, we are able to regain previously known special solutions. More qualitative and quantitative results can be obtained using our framework, and will be published elsewhere. In Part II, we shall investigate a perturbative approach to the system by taking ε to be a small number.

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