GENERAL TREATMENT OF GEODETIC AND LENSE–THIRRING EFFECTS ON AN ORBITING GYROSCOPE

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Abstract

We give a general derivation of the metric of a spinning body of any shape and composition using linearized general relativity theory (LGRT), and also obtain the same metric using a simple transformation argument. The latter derivation makes it clear that the linearized metric contains only the Eddington γ and $\alpha\,(\equiv 1)$ parameters, so no new parameter is involved in any frame–dragging or Lense–Thirring (LT) effects. We then calculate the precession of an orbiting gyroscope in a general gravitational field, described by a Newtonian potential (gravito-electric field) and a vector potential (gravito-magnetic field). Finally we do a multipole analysis and give the general spherical harmonics expansion of the precession in terms of multipoles of the scalar and vector potentials, i. e., moments of the density distribution. In particular, in regard to the Gravity Probe B (GP-B) experiment, we find that the effect of the Earth's quadrupole moment J_2 on the geodetic precession is large enough to be measured by GP-B (a previously known result), but the effect on the LT precession is somewhat beyond the expected GP-B accuracy.

1 Introduction

The Gravity Probe B satellite is scheduled to fly in the year 2000 [1]. It contains a set of gyroscopes intended to test the predictions of general relativity (GR) that a gyroscope in a low (altitude$\approx 650\,km$) circular polar orbit will precess about 6.6 *arcsec/year* in the orbital plane (geodetic precession) and about 42 *milliarcsec/year* perpendicular to the orbital plane (LT precession, see [2]; [3], secs. 4.7 and 7.8; [4], sec. 9.1). In this paper we review the theoretical derivation of these effects and in particular consider the Earth's quadrupole and higher multipole fields' contribution to them.

We first review the derivation of the metric for a rotating body using the standard LGRT approach ([5]; [2]; [3], secs. 4.7 and 7.8). The metric is characterized by a Newtonian scalar potential (called the gravito-electric field) and a vector potential (called the gravito-magnetic field) ([3], secs. 3.5). We then obtain the same result with a simple transformation argument which clarifies the physical meaning of the metric ([4], sec. 4.3). Specifically it makes clear that if the metric of a point mass contains fundamental parameters such as the Eddington parameters α and γ, then to lowest order the metric of a rotating body contains no **new** fundamental parameters [6]. Thus there is no new "LT parameter" to be measured by GP-B – or any other experiment.

We then derive the precession equations for a gyroscope in a general way, that is for any scalar and vector potential fields [2]. The calculation is valid to first order in the fields and velocities of the source body and the satellite. The gravitational field of the earth is described by the scalar and vector potentials which depend on the shape of the body and the mass distribution inside it; in addition, the gravito-magnetic potential depends also on the rotation velocity ([3], sec. 3.5). We treat both of these fields by a multipole expansion and express the precessions in series of spherical harmonics whose coefficients are combinations of scalar and vector potential multipoles, in other words, of spherical harmonics moments of the density distribution. In particular we show that up to the order $l \leq 2$, both precessions depend only on the tensor of inertia of the earth. The major contributions to the GP-B precessions are from the Earth's quadrupole moment and both have a magnitude of about 1 part in 10^3. The contribution to the geodetic precession is detectable by GP-B and quite important for the determination of the parameter γ which is to be measured to about 1 part in 10^5, the most accurate measurement envisioned ([3], sec. 3.5 and in particular table 14.2; [7]); the contribution to the LT precession is somewhat beyond the GP-B accuracy.

2 The Lense–Thirring Metric and Eddington Parameters

2.1 Derivation by Linearized General Relativity

This is a standard derivation, so we review it briefly [2].

The metric of a rotating body such as the Earth in LGRT is obtained by introducing a small perturbation $h_{\mu\nu}$ of the Lorentz metric $\eta_{\mu\nu}$, that is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. The perturbation is assumed to be independent of time and isotropic in space, $h_{11} = h_{22} = h_{33} = h_s$. The energy–momentum tensor of slow-moving and low density matter with negligible pressure is $T^{\mu\nu} = \rho u^\mu u^\nu$, where u^μ is the 4–velocity and ρ is the matter density, the field equations are then ($R_{\mu\nu}$ is the Ricci tensor):

$$R_{\mu\nu} = -8\pi G\big(T_{\mu\nu} - (1/2)g_{\mu\nu}T\big), \qquad T = T^{\sigma}_{\sigma} \tag{1}$$

The calculation of $R_{\mu\nu}$ and $T_{\mu\nu}$ to the lowest order in the perturbation is straightforward and results in the following form of equations (1):

$$\begin{aligned} \Delta\big(h_{00}/2\big) = \Delta\big(h_s/2\big) = 4\pi G\rho \\ \Delta h_{0i} - h_{0l|li} = 16\pi G\rho\, v^i \end{aligned} \tag{2}$$

In the last expression we have kept only first order terms in the velocity in the off-diagonal metric elements. The first of equations (2) is the Poisson equation for the metric perturbation, so by correspondence with the classical theory the latter should be related to the classical gravitational potential by the well-known equality [2]:

$$h_{00} = h_s = 2\Phi \tag{3}$$

Equations (2) may be solved using the Green function as ($\vec{h} = \{h_{01},\, h_{02},\, h_{03}\}$)

$$\Phi(\vec{r}) = -G\int \frac{\rho(\vec{r}\,')\, d^3\vec{r}\,'}{|\vec{r} - \vec{r}\,'|}, \qquad \vec{h}(\vec{r}) = 4G\int \frac{\rho(\vec{r}\,')\vec{v}(\vec{r}\,')\, d^3\vec{r}\,'}{|\vec{r} - \vec{r}\,'|} \tag{4}$$

Note that these expressions are analogs of the equations of electrostatics and magnetostatics, which is why one may speak about gravitoelectric and gravitomagnetic effects in LGRT described by gravitoelectric potential (Φ) and gravitomagnetic vector potential $\vec{h}$ [2]. In summary, we may write the Lense–Thirring line element as

$$ds^2 = \big(1 + 2\Phi\big)dt^2 - \big(1 - 2\Phi\big)d\vec{r}^{\,2} + 2\vec{h}\cdot d\vec{r}\,dt \tag{5}$$

2.2 Derivation by Transformation; Eddington Parameters

It is possible to obtain the above result from a different and physically interesting perspective, and moreover introduce parameters convenient for discussing experimental measurements. Following Eddington, consider the metric of a massive point with a geometric mass m at a large distance r, $m/r \ll 1$. Expand the Schwarzschild solution in isotropic coordinates for this situation as

$$ds^2 = \frac{(1 - m/2r)^2}{(1 + m/2r)^2}dt^2 - (1 + m/2r)^4 d\vec{r}^{\,2}$$

$$= \left(1 - \frac{2m}{r} + \frac{2m^2}{r^2} + \ldots\right)dt^2 - \left(1 + \frac{2m}{r} + \frac{3m^2}{r^2} + \ldots\right)d\vec{r}^{\,2}$$

Eddington suggested that this be written in terms of parameters as [8], [9]

$$ds^2 = \left(1 - \alpha\frac{2m}{r} + \beta\frac{2m^2}{r^2} + \ldots\right)dt^2 - \left(1 + \gamma\frac{2m}{r} + \ldots\right)d\vec{r}^{\,2}; \tag{6}$$

The Eddington parameters α, β and γ are equal to 1 for GR. The power series (5) is clearly a rather general form for the metric far from a spherical body. Since the m which appears in the metric (6) is merely a constant of integration representing the mass of the central body (specifically $m = GM/c^2$), we may absorb parameter α into it, which is equivalent to taking $\alpha \equiv 1$. This is consistent as long as no independnt nongravitational determination of the mass of the central body (the earth) is considered. We shll nevertheless display the α in our calculations as a book-keeping device.

The parameters may be used as a tool for tracking the terms in the metric which contribute to some gravitational effect; alternatively they may be viewed as numbers which may **not** be equal to 1 if a metric theory other than GR is actually valid; in either case they provide a convenient way to express the results of experimental tests of gravity as giving value to parameters. This parametrized approach has been extended to include many other parameters and has been highly developed under the name parametrized post–Newtonian theory, or PPN [4]. In this paper we take the viewpoint that GR is to be tested and emphasize that we are **not** using the more general PPN approach.

We consider only phenomena in which the term $\sim m^2/r^2$ in g_{00} of (6) is unimportant, that is in which we may ignore β and thus have an underlying linear theory. Then, for a *stationary* mass,

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2} \tag{7}$$

Since this is nearly the Lorentz metric, we may generalize it to a *moving* mass point by simply using the frame transformation that is Lorentzian to the first order in velocity: $t_r = t - vx, \quad x_r = x - vt$; here the subscript r labels the system in which the mass is at rest, and which moves at v in the positive x direction. This gives the metric for the *moving* point mass as

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2} + (\alpha + \gamma)\frac{4m}{r}\,v dx\, dt,$$

which obviously generalizes for motion in any direction to

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2} + (\alpha + \gamma)\frac{4m}{r}\,(\vec{v}\cdot d\vec{r})\, dt \tag{8}$$

Since we assume that our theory is linear to this order, we can superpose the fields of any distribution of such point masses and write $\Phi(\vec{r})$ of (4) in place of m/r and $\vec{h}(\vec{r})$ of (4) in place of $4mG\vec{v}/r$, resulting in

$$ds^2 = (1+2\alpha\Phi)\,dt^2 - (1-2\gamma\Phi)\,d\vec{r}^{\,2} + (\alpha+\gamma)\,(\vec{h}\cdot d\vec{r})dt \tag{9}$$

This agrees with the LGRT formula (5) when $\alpha = \gamma = 1$ but now contains appropriate combinations of Eddington parameters. We emphasize that line element (9) has been obtained for any slowly moving mass distribution from the parametrized metric for a stationary mass point by transformation and superposition, and thus no new parameter appears in its expression. Therefore a measurement of a phenomenon which depends on the cross term in (9) provides a value for $\alpha + \gamma$ and **not** for some new parameter, i. e., **does not** provide an independent test of gravitational theory [6].

3 Precession Equations

An orbiting gyroscope has its spin axis parallel displaced in accord with metric (9), that is ([2]; [3], secs. 4.7 and 7.8; [4], sec. 9.1):

$$\frac{dS^\mu}{ds} + \Gamma^\mu_{\nu\sigma} S^\nu \frac{dx^\sigma}{ds} = 0 \tag{10}$$

Assuming that the gyro spin 4-vector is perpendicular to the velocity 4-vector (that is the spin has no zero component it its rest frame), we work to the first order in the potential and the velocities, $\vec{v}$ and $\vec{V}$, of the central body and orbiting gyro, respectively. We calculate the Christoffel symbols from metric (9) and use the corresponding Euler - Lagrange equations of motion to convert (10) into the final 3D equation of spin evolution with antisymmetric and symmetric parts separated:

$$\frac{d\vec{S}}{dt} = \vec{\Omega}\times\vec{S} - \frac{1}{2}\Big\{2\gamma(\vec{V}\cdot\nabla\Phi)\vec{S} - \alpha\Big[(\vec{S}\cdot\vec{V})\nabla\Phi + (\vec{S}\cdot\nabla\Phi)\vec{V}\Big]\Big\} \tag{11}$$

Here $\vec{\Omega} = \vec{\Omega}_G + \vec{\Omega}_{LT}$ is the sum of the geodetic and Lense–Thirring precessions given by

$$\vec{\Omega}_G = \left(\frac{\alpha+2\gamma}{2}\right)\nabla\Phi\times\vec{V}, \qquad \vec{\Omega}_{LT} = \left(\frac{\alpha+\gamma}{2}\right)\frac{1}{2}\nabla\times\vec{h}, \tag{12}$$

hereas the second term is responsible for stretching of $\vec{S}$, which effect is extremely 1all under the conditions similar to GP-B. When averaged over the closed orbit, this term gives a zero contribution, as found from the motion equation $\nabla\Phi = -\dot{\vec{V}}$ (c. f. $\nabla\Phi \cdot \dot{\vec{V}} = 0.5\, dV^2\,/dt$, etc.); in reality, a small orbit variation might produce some very small average which is inversely proportional to the time and may be neglected, anyway. Therefore the average drift rate of the gyro spin is

$$\langle\dot{\vec{S}}\rangle = \dot{\vec{S}}_G + \dot{\vec{S}}_{LT}, \qquad \dot{\vec{S}}_A = \langle\vec{\Omega}_A\rangle \times \vec{S}, \quad A = G,\ LT, \tag{13}$$

with the averaged values of precessions found from (12).

4 Effect of Distant Masses

When some distant masses M_n, such as the moon, move with velocities v_n relative to the central body, the density distribution $\rho(\vec{r})$ may be expressed as

$$\rho_N(\vec{r}) = \rho(\vec{r}) + \sum_{n=1}^{N} M_n \delta(\vec{r} - \vec{r}_n)$$

In their turn, the scalar and vector potentials (4) are now replaced by

$$\Phi_N(\vec{r}) = \Phi(\vec{r}) - G\sum_{n=1}^{N} \frac{M_n}{|\vec{r} - \vec{r}_n|}, \qquad \vec{h}_N(\vec{r}) = \vec{h}(\vec{r}) + 4G\sum_{n=1}^{N} \frac{M_n \vec{v}_n}{|\vec{r} - \vec{r}_n|}$$

From (12) we find, in particular, the LT precession including the effect of distant masses ($\vec{L}_n = M_n(\vec{r} - \vec{r}_n) \times \vec{v}_n$ is the angular momentum):

$$\vec{\Omega}^N_{LT} = \vec{\Omega}_{LT} - (\alpha + \gamma)G\sum_{n=1}^{N} \frac{\vec{L}_n}{|\vec{r} - \vec{r}_n|^3},$$

5 Solid Body Rotation: Multipole Expansions

5.1 Solid Body Rotation: Vector Potential $\Pi(\vec{r})$

From now on, we assume, for brevity, the GR values $\alpha = \gamma = 1$, and study the case of rigid rotation: $\vec{v} = \vec{\omega} \times \vec{r}$. We see from (4) that it is natural to introduce a new vector potential $\vec{\Pi}(\vec{r})$ by setting

$$\vec{h}(\vec{r}) = 4\omega \times \vec{\Pi}(\vec{r}), \qquad \vec{\Pi}(\vec{r}) = \int \frac{\rho(\vec{r}\,')\vec{r}\,'\, d^3\vec{r}\,'}{|\vec{r} - \vec{r}\,'|} \tag{14}$$

From (4) and (14) we derive the important relation $\nabla \cdot \vec{\Pi} = -(\Phi + \vec{r} \cdot \nabla\Phi)$ which allows us to rewrite expression (4) for $\vec{\Omega}_{LT}$ as

$$\vec{\Omega}_{LT} = 2\left[-(\vec{\omega} \cdot \nabla)\vec{\Pi} + \vec{\omega}\Big(\nabla \cdot \vec{\Pi}\Big)\right] = -2\left[(\vec{\omega} \cdot \nabla)\vec{\Pi} + \vec{\omega}\Big(\Phi + \vec{r} \cdot \nabla\Phi\Big)\right], \qquad (15)$$

or, choosing the z axis along $\vec{\omega}$,

$$\vec{\Omega}_{LT} = -2\omega\left[\vec{\Pi}_{,z} + \Big(\Phi + r\Phi_{,r}\Big)\right]\hat{e}_3, \qquad \vec{\omega} = \omega\,\hat{z} \qquad (16)$$

Note also that the metric (5) in terms of $\Pi(\vec{r})$ is:

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)d\vec{r}^2 + 8(\vec{\omega} \times \vec{\Pi}) \cdot d\vec{r}\,dt \qquad (17)$$

5.2 Multipole Expansions: General

We choose the origin of spherical coordinates $\{r, \theta, \varphi\}$ at the center of mass of the earth, introduce spherical harmonics

$$Y^{\nu}_{lm}(\theta, \varphi) = P^m_l(\cos\theta)\begin{cases}\cos m\varphi, & \nu = c \\ \sin m\varphi, & \nu = s\end{cases},$$

and expand the potentials Φ and $\vec{\Pi}$ in corresponding series:

$$\Phi(\vec{r}) = -\frac{GM}{r}\left[1 + \sum_{l\geq 2,\, m,\, \nu} a^{\nu}_{lm}\left(\frac{R}{r}\right)^l Y^{\nu}_{lm}\right], \quad \Pi_i(\vec{r}) = \frac{GMR}{r}\sum_{l\geq 1,\, m,\, \nu} p^{i\nu}_{lm}\left(\frac{R}{r}\right)^l Y^{\nu}_{lm} \qquad (18)$$

Here R is the characteristic size of the body (we will use the equatorial radius for the Earth), and the coefficients are related to the mass distribution by

$$a^{\nu}_{lm} = \frac{(2 - \delta_{m0})}{M}\frac{(l - m)!}{(l + m)!}\int \rho(\vec{r})\left(\frac{r}{R}\right)^l Y^{\nu}_{lm}\, d^3\vec{r}, \quad m = 0, 1, \ldots, l; \qquad (19)$$

$$p^{i\nu}_{lm} = \frac{(2 - \delta_{m0})}{M}\frac{(l - m)!}{(l + m)!}\int \rho(\vec{r})\left(\frac{r}{R}\right)^l \frac{x_i}{R} Y^{\nu}_{lm}\, d^3\vec{r}, \quad \nu = c, s;\ i = 1, 2, 3 \qquad (20)$$

(For convenience, we shall write $a^c_{l0} \equiv a_{l0}$, $a^s_{l0} \equiv 0$ $p^{ic}_{l0} \equiv p^i_{l0}$, $p^{is}_{l0} \equiv 0$). Among many reasons for doing the multipole expansions, one is that the gravity coefficients a^{ν}_{lm} are very well measured for the Earth at least up to $l = 18$ ([10])

From (12) and (16) the multipole expansions for $\vec{\Omega}_G$ and $\vec{\Omega}_{LT}$ may be computed; the first calculation is straightforward since it reduces to finding $\nabla\Phi$; the second one would be rather difficult but for expression (16) which simplifies it significantly. Thus, with the z axis along $\vec{\omega}$, we have a surprisingly simple expansion for $\vec{\Omega}_{LT}$:

$$\Omega^i_{LT} = \frac{GM\omega}{r} \sum_{l\geq 2,\, m,\, \nu} \left[(l-m)\, p^{i\nu}_{l-1\,m} - l\, a^{\nu}_{lm}\delta_{i3}\right] \left(\frac{R}{r}\right)^l Y^{\nu}_{lm}, \quad \vec{\omega} = \omega\,\hat{z} \tag{21}$$

For any frame rotated from this one the corresponding expansion is found in the standard way by means of rotation matrices (see [11]).

For the earth of any shape and mass distribution it is impossible to express $p^{i\nu}_{lm}$ through a^{ν}_{lm}, in other words, the values of the scalar and vector potentials are independent. Nevertheless, a useful relationship between two sets of coefficients exists; to describe it, we need a notation for a general moment of the density,

$$M^{\nu}_{klm} \equiv \int \rho(\vec{r}) \left(\frac{r}{R}\right)^k Y^{\nu}_{lm}\, d^3\vec{r}; \tag{22}$$

in particular,

$$a^{\nu}_{lm} = \frac{(2-\delta_{m0})}{M} \frac{(l-m)!}{(l+m)!} M^{\nu}_{llm} \tag{23}$$

Using definitions (20) and (21) and the recurrence realtions for Legendre functions [14], we derive the following equalities relating $p^{i\nu}_{lm}$ to a^{ν}_{lm}:

$$p^{1\nu}_{lm} = (2l+1)^{-1}\Big\{-2^{-1}(l+m+1)(l+m+2)a^{\nu}_{l+1m+1} + (2-\delta_{m1})^{-1}a^{\nu}_{l+1m-1} +$$

$$\Big[(l-m)!/M(l+m)!\Big]\left[M^{\nu}_{l+1l-1m+1} - (l+m-1)(l+m)M^{\nu}_{l+1l-1m-1}\right]\Big\}$$

$$p^{2\nu}_{lm} = (\mp)(2l+1)^{-1}\Big\{2^{-1}(l+m+1)(l+m+2)a^{\mu}_{l+1m+1} + (2-\delta_{m1})^{-1}a^{\mu}_{l+1m-1} -$$

$$\Big[(l-m)!/M(l+m)!\Big]\left[M^{\mu}_{l+1l-1m+1} - (l+m-1)(l+m)M^{\mu}_{l+1l-1m-1}\right]\Big\} \tag{24}$$

$$p^{3\nu}_{lm} = (2l+1)^{-1}\Big\{(l+m+1)a^{\nu}_{l+1m+1} + (2-\delta_{m0})(l-m)!M^{\nu}_{l+1l-1m}/M(l+m-1)!\Big\}$$

In the second line of (24), the minus sign is taken and $\mu = s$ when $\nu = c$, the plus sign and $\mu = c$ when $\nu = s$.

5.3 Multipole Expansions for $l \leq 2$ and the Inertia Tensor

If the shape of the central body and the mass distribution inside it are known, then all the pertinent quantities, including multipole expansion coefficients a^{ν}_{lm} and $p^{i\nu}_{lm}$, may be found by integration, but this is rarely the case. Even when a^{ν}_{lm} are measured, as for the Earth, all the $p^{i\nu}_{lm}$, and the LT effect with them, remain entirely undetermined. However, for a body of any shape and composition, a^{ν}_{2m}, $(l = 2)$ and $p^{i\nu}_{1m}$, $(l = 1)$ can be expressed in terms of elements $I_{ij} = \int \rho(\vec{r})\,(r^2\delta_{ij} - x_i x_j)\,d^3\vec{r}$ of the tensor of inertia $\mathbf{I}$ (we write $I_{ii} \equiv I_i$):

$$\begin{gathered}
a_{20} = -(2I_3 - I_2 - I_1)/(2MR^2), \qquad a^c_{22} = (I_2 - I_1)/(4MR^2) \\
p^3_{10} = -(I_3 - I_2 - I_1)/(2MR^2), \qquad p^{1c}_{11} = -(I_3 + I_2 - I_1)/(2MR^2) \\
p^{2s}_{11} = -(I_3 - I_2 + I_1)/(2MR^2) \\
a^c_{21} = -p^1_{10} = p^{3c}_{11} = I_{13}/(MR^2), \quad a^s_{21} = -p^2_{10} = p^{3s}_{11} = I_{23}/(MR^2) \\
-a^s_{22} = p^{1s}_{11}/2 = p^{2c}_{11}/2 = I_{12}/(MR^2)
\end{gathered} \tag{25}$$

This is done by comparing the integrals (18) (with $l = 2$) and (19) (with $l = 1$) to I_{ij} using explicit expressions of Legendre functions with $l = 1, 2$; formulas for a_{20} and a^c_{22} are known and used in geodesy for the determination of Earth's moments of inertia ([12]).

Introducing (25) into (17) and dropping the terms with $l > 2$ for Φ and $l > 1$ for $\vec{\Pi}$, we first obtain the $l \leq 2$ formulas for the potentials,

$$\Phi(\vec{r}) = -\frac{G}{r}\left\{M + \frac{1}{2r^2}\left[\mathrm{tr}\,\mathbf{I} - \frac{3}{r^2}\,(\mathbf{I}\vec{r}\cdot\vec{r})\right]\right\}, \quad \vec{\Pi}(\vec{r}) = -\frac{G}{r^3}\left[\mathbf{I}\vec{r} - \frac{1}{2}\,(\mathrm{tr}\,\mathbf{I})\vec{r}\right] \tag{26}$$

and then from those, by doing differentiation in (12), we obtain for the precessions:

$$\begin{aligned}
\vec{\Omega}_G &= \tfrac{3G}{2r^3}\left\{\left[M + \tfrac{3}{2r^2}\left(\mathrm{tr}\,\mathbf{I} - \tfrac{5}{r^2}\,(\mathbf{I}\vec{r}\cdot\vec{r})\right)\right](\vec{r}\times\vec{V}) + \tfrac{3}{r^2}\,(\mathbf{I}\vec{r}\times\vec{V})\right\} \\
\vec{\Omega}_{LT} &= \tfrac{2G}{r^3}\left\{\mathbf{I}\vec{\omega} - 3\left[\tfrac{1}{2}\,(\mathrm{tr}\,\mathbf{I}) - \tfrac{1}{r^2}\,(\mathbf{I}\vec{r}\cdot\vec{r})\right]\vec{\omega} + 3\,\tfrac{\vec{\omega}\cdot\vec{r}}{r^2}\left[\tfrac{1}{2}\,(\mathrm{tr}\,\mathbf{I})\vec{r} - \mathbf{I}\vec{r}\right]\right\}
\end{aligned} \tag{27}$$

(Of course, the same expression for $\vec{\Omega}_{LT}$ is also obtained from (25) and (21) with $l = 2$). These expressions are valid under either of two conditions: 1) *far field*, $R/r \ll 1$; 2) *high symmetry* (all higher order moments are small). They alter our notion of the geodetic and Lense–Thirring effects: the first one is proportional not only to the orbital momentum, but to $\mathbf{I}\vec{r}\times\vec{V}$ as well, and the LT precession points not in the direction of the angular momentum $\vec{L} = \mathbf{I}\vec{\omega}$, but has also components parallel to $\vec{\omega}$, $\vec{r}$, and $\mathbf{I}\vec{r}$. Two particular cases of the inertia tensor are of special interest.

a) Spherical symmetry, $\mathbf{I} = \mathrm{diag}\{I, I, I\}$. In this case, the classical Lense–Thirring formula follows immediately from (27):

$$\vec{\Omega}_{LT} = \frac{2GI}{r^3}\left[-\vec{\omega} + \frac{3}{r^2}(\vec{\omega}\cdot\vec{r})\vec{r}\right] \tag{28}$$

Note that we have thus shown this to be the exact result for a spherical earth with any *radial* density distribution $\rho = \rho(r)$.

b) Symmetric top, $\mathbf{I} = \mathrm{diag}\{I_1, I_1, I\}$, $I_1 \neq I$. For $\vec{\omega} = \omega\hat{z}$, the previous expression remains true; to the main order in the oblateness, this proves to be the exact result for a slightly oblate uniform ellipsoid of revolution rotating about its semiminor axis.

6 Earth Models and Results for GP-B

6.1 Earth Models

To go beyond the $l \leq 2$ approximation, one must make some assumptions about the shape of the earth and density distribution inside it, and use the data of gravitational potential measurements, if available. In the case of the Earth and GP-B conditions ($R/r \approx 0.9$), one needs to check the validity of the $l \leq 2$ approximation to ensure that theoretical predictions match the expected experimental accuracy (10^{-5} for geodetic effect, $\sim 10^{-2}$ for LT effect). Here is our set of reasonable assumptions.

1.Gravitational potential For the required accuracy, it is enough to include only the quadrupole moment into Φ, i. e., to set

$$-a_{20} \equiv J_2 \approx 1.083 \times 10^{-3}; \qquad a^{\nu}_{lm} = 0 \quad \text{for} \quad l = 2, m > 0; \quad l > 2, \tag{29}$$

because all gravitational coefficients other than the Earth's oblateness J_2 are at least 2 orders of magnitude smaller [10]. Then, by (18) and (25), the $l \leq 2$ expression (26) for Φ is valid with $\mathbf{I} = \mathrm{diag}\{I_1, I_1, I\}$, $I_1 = I - J_2MR^2$. Also, by (12), (26) and (29), the geodetic precession is completely determined including the J_2 correction calculated for the first time by Breakwell [7]. Note that small and/or slow motion of the Earth rotation axis relative to the Earth centered inertially fixed frame may be neglected.

2.Shape It is sufficient to assume that the Earth is a slightly oblate ellipsoid of revolution (the Clairaut formula, see Roy [13]), so that to the first order in eccentricity ϵ ($\approx 3.353 \times 10^{-3}$) the surface equation is $r = r_s(\theta, \varphi) = R(1 - \epsilon \cos^2 \theta)$.

3.Mass distribution We examine two different models:

a) With ρ_0, $\Delta\rho$ being arbitrary functions of their arguments, set

$$\rho(\vec{r}) = \rho_0(r) + \Delta\rho(\theta, \varphi), \qquad \int_{\text{unit sphere}} \Delta\rho(\theta, \varphi) \sin\theta \, d\theta d\varphi = 0 \tag{30}$$

The first term here describes any depth variation of the *average* density, and the only assumption is that the angular variations are depth–independent.

b) For arbitrary ρ_0 and ρ_s, set

$$\rho(\vec{r}) = \rho_0(r) + \rho_s(\theta, \varphi)\delta(r - r_s(\theta, \varphi)) \tag{31}$$

Contrary to the previous model, here all angular variations of the density are concentrated at the Earth's surface, which is rather realistic, since the estimated thickness of the layer where the mass distribution varies significantly in the angular directions is about $30\,km$.

Note that instead of 3a) and 3b) an entirely different assumption is used in geodesy, namely, that the sum of gravitational and centrifugal potentials is constant at the Earth's ellipsoid surface ([15]). It allows one to relate a_{l0} to the eccentricity and the Earth angular velocity only (in particular, to obtain J_2 with a surprisingly good accuracy), but gives zero values to a^{ν}_{lm}, $m \neq 0$ and leaves $p^{i\nu}_{lm}$ undetermined.

Evidently, our two models should give the bounds for the corrections to the $l \geq 2$ values of the LT effect. Moreover, for both of them it is possible to find the corrections explicitly by means of the following procedure. Working all the way to the first order in the eccentricity, we 1) using (19), fit (30), model a), (or (31), model b),) to the values of a^{ν}_{lm}; this permits one to determine completely the function $\Delta\rho(\vartheta, \varphi)$ (respectively, $\rho_s(\theta, \varphi)$) and all the moments M^{ν}_{klm}; 2) using the latter, calculate $p^{i\nu}_{lm}$ for $l > 1$ by (24), determining thus the LT precession by (21). It turns out that *for both models* the only additional non-zero coefficients are those with $l = 3$, $m = 0, 1$ under condition (26); their values for model a) are

$$p^{1c}_{31} = p^{2s}_{31} = (5/49)\, J_2; \qquad p^{3}_{l0} = -(15/49)\, J_2; \tag{32}$$

for model b) they are larger by just the factor $7/5$.

Let us show briefly the appropriate derivation using model a) as an example; calculations for model b) are absolutely similar. First, we introduce (30) into the definition (22) of the general moment of density, and working to the main order in eccentricity ϵ, find it in terms of Fourier coefficients $\rho^{\nu}_{lm} = \int \Delta\rho Y^{\nu}_{lm} \sin\theta d\theta d\varphi$ of function $\Delta\rho(\theta,\varphi)$:

$$M^{\nu}_{klm} = R^3(k+3)^{-1}\Big\{[1-\epsilon(k+3)V_{lm}]\rho^{\nu}_{lm}-\epsilon(k+3)[S_{lm}\rho^{\nu}_{l+2,m}+T_{lm}\rho^{\nu}_{l-2,m}]\Big\}+O((\epsilon k)^2) \tag{33}$$

Here V_{lm}, S_{lm}, T_{lm} are known positive rational fractions of l and m bounded for all their values, and the formula is lightly different for the case $l = m = 0$, but we do not need it. For $k = l$ the left-hand side of this equality is given via a^{ν}_{lm} according to (23), thus by inverting this tri-diagonal system for ρ^{ν}_{lm} with small off-diagonal elements, we express the latter in terms of a^{ν}_{lm} and then, iserting the result back into (33), we get *all* moments expressed through the gravitational coefficients:

$$M^{\nu}_{klm} = \frac{M}{2-\delta_{m0}}\left\{\frac{(l-m)!}{(l+m)!}\frac{l+3}{k+3}a^{\nu}_{lm} - \right. \tag{34}$$

$$\left. \epsilon\frac{k+2}{k+3}\left[\tilde{V}_{lm}a^{\nu}_{lm} + \tilde{S}_{lm}a^{\nu}_{l+2,m} + \tilde{T}_{lm}a^{\nu}_{l-2,m}\right]\right\} + O((\epsilon l)^2);$$

quantities $\tilde{V}_{lm}$, $\tilde{S}_{lm}$, $\tilde{T}_{lm}$ are simply related to V_{lm}, S_{lm}, T_{lm}, respectively. By (24), we need only $k = l \pm 1$ in (34) to obtain $p^{i\nu}_{lm}$ in terms of a^{ν}_{lm}, and the result is:

$$p^{1\nu}_{lm} = \frac{1}{2l+1}\left[-\frac{(l+m+1)(l+m+2)}{2}a^{\nu}_{l+1m+1} + \frac{(l-m)(l-m-1)(l+2)}{2(l+4)}a^{\nu}_{l-1m+1} + \right.$$

$$\left. \frac{1}{2-\delta_{m1}}\left(a^{\nu}_{l+1m-1} - \frac{l+2}{l+4}a^{\nu}_{l-1m-1}\right)\right] + O((\epsilon l)^2)$$

$$p^{2\nu}_{lm} = \frac{(\mp)}{2l+1}\left[\frac{(l+m+1)(l+m+2)}{2}a^{\mu}_{l+1m+1} + \frac{(l-m)(l-m-1)(l+2)}{2(l+4)}a^{\mu}_{l-1m+1} - \right.$$

$$\left. \frac{1}{2-\delta_{m1}}\left(a^{\mu}_{l+1m-1} + \frac{l+2}{l+4}a^{\mu}_{l-1m-1}\right)\right] + O((\epsilon l)^2) \tag{35}$$

$$p^{3\nu}_{lm} = \frac{1}{2l+1}\left[(l+m+1)a^{\nu}_{l+1m} + \frac{(l-m)(l+2)}{(l+4)}a^{\nu}_{l-1m}\right] + O((\epsilon l)^2),$$

with the minus and $\mu = s$ for $\nu = c$, plus and $\mu = c$ for $\nu = s$ in the second of these formulas. Whatever values of the gravitational coefficients are specified (by measurement), $p^{i\nu}_{lm}$, and therefore the vector potential (18) and the LT precession (21), are found by (35) under the above assumptions 2) and 3a). If, in addition, equality (29), that is assumption 1), is valid, then that gives exactly the result (32).

6.2 Results for GP-B

Using the above results, let us finally give the values of geodetic and LT precessions averaged over the circular polar orbit of 650 km altitude perturbed by the quadrupole moment. To the main order in the oblateness J_2,

$$\left\langle \Omega_G \right\rangle = \frac{3GMV}{2r^2}\left[1-\frac{9}{8}J_2\left(\frac{R}{r_0}\right)^2\right]$$

$$\left\langle \Omega_{LT} \right\rangle = \frac{GI\omega}{2r^3}\left[1+\frac{9}{8}J_2\left(\frac{R}{r_0}\right)^2\left(1-\frac{88}{147}\frac{MR^2}{I}\right)\right]$$

where the values (29) for model a) are used. The corrections in both cases are close to 0.1%, which is beyond GP-B accuracy for the LT effect but two orders of magnitude larger than the expected measurement error for geodetic effect. Since GP-B is intended to measure the geodetic precession and parameter γ to about a part in 10^5, this is a critically important correction.

Acknowledgements

This work was supported by NASA grant NAS 8-39225 to Gravity Probe B. We are grateful to the members of GP-B Theory Group, especially, to C.W.F.Everitt, G.M.Keiser and R.V.Wagoner, for many fruitful discussions and enlightening comments.

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