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Metric for an Oblate Earth

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In linearized general relativity the metric of a body is described by a scalar potential and a three-vector potential. We here present a simple transformation derivation of the linearized metric in terms of these potentials, and calculate the exact scalar and vector potentials for a field with oblate spheroidal symmetry. The results for the external potentials do not depend on details of the density distribution inside the earth; both the scalar and vector potentials are fully determined by the total mass, the total angular momentum, and a radial parameter, all of which are accurately known from observation. The scalar potential is accurate to roughly 10^{-6} and the vector potential, which has never been accurately measured, should be accurate to about 10^{-5}. Applications include an accurate treatment of the details of the motion of satellites, and the precession of a gyroscope in earth orbit.

KEY WORDS : General relativity ; gravity ; gravitomagnetism ; oblate earth

1. INTRODUCTION

Soon after the discovery of general relativity Lense and Thirring [1] gave an approximate linearized analysis in which the metric is expressed in terms of a scalar potential, which is the same as the classical Newtonian potential, and a three-vector potential, which is the analog of the vector potential of classical electrodynamics (Ref. 2, Ch. 3 and 4, Ref. 3, sec. 40.7, Ref. 4,

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sec. 9.2). The same result may be obtained by using a transformation argument, which clarifies its physical meaning [5–8]. We briefly review the transformation derivation because of its importance in the experimental testing of general relativity, and because it does not appear to be widely known.

The Newtonian scalar potential of the earth has approximately the symmetry of an oblate ellipse of revolution, that is oblate spheroidal symmetry (Ref. 9, sec. 1.3, Ref. 10, p.546 and sec. 2.11, Ref. 11). We obtain the exact scalar and vector potentials, within the context of linearized general relativity, of a field with this symmetry, exterior to the earth. Our derivation uses oblate spheroidal coordinates, and is a generalization of the derivation of the potentials for a sphere.

The mathematical problem of the scalar potential is the same as the electrostatic problem of a charged conducting oblate spheroid, and the solution to that problem may be found in the literature (Ref. 12, p.124, sec. 5.271, p.292 and p.322, and Ref. 13, sec. 1.4). Vinti noted the usefulness of this solution in the context of classical gravitational theory and satellite geodesy [11]. The mathematical problem of the vector potential is the same as the magnetostatic problem of an oblate spheroid with a surface current; a problem of this general type is noted by Smythe [12], and Jackson and Durand discuss the special case of a charged rotating sphere (Ref. 14, p.166, and Ref. 15). We know of no discussion of the solution presented here.

We include a short section on the multipole expansion of both potentials, and another on the relation of the exterior field and the shape of the earth's surface [7,9]. These provide a numerical evaluation of the radial parameter which appears in the spheroidal coordinate system, and a consistency check and estimate of the accuracy of the fields.

The scalar potential field of the earth has been accurately measured by the use of satellite geodesy [9,16]. Rough evidence for the vector field is provided by observations of the Lageos satellites, but the accuracy is low and controversial [17]. The only presently viable method for an accurate measurement is the precession of the gyroscope on the Gravity Probe B satellite, which is scheduled for launch in the year 2000 [18]. The calculation of the precession of such a gyroscope in earth orbit will be carried out with the use of our solution in another work.

2. THE METRIC

Within the context of linearized general relativity theory Lense and Thirring [1] obtained the general form for the metric for a time independent weak field system. The metric, in isotropic spatial coordinates, is (with $c = 1$)

$$ds^2 = (1 + 2\phi)dt^2 - (1 - 2\phi)d\vec{r}^{\,2} + 2(\vec{h} \cdot d\vec{r})dt. \tag{1}$$

Here ϕ is the Newtonian potential and $\vec{h}$ is a gravitational three-vector potential, analogous to the vector potential of classical electrodynamics. The potentials are determined in terms of the mass density ρ and velocity $\vec{v}$ of the source material by the following equations [2,3]:

$$\nabla^2\phi = 4\pi G\,\rho, \tag{2a}$$

$$\nabla^2\vec{h} - \nabla(\nabla \cdot \vec{h}\,) = -16\pi G\,\rho\vec{v}. \tag{2b}$$

These have Green's function solutions

$$\phi(\vec{r}\,) = -G\int \frac{\rho(\vec{r}\,')\,d^3r'}{|\vec{r} - \vec{r}\,'|}, \tag{3a}$$

$$\vec{h}\,(\vec{r}\,) = -G\int \frac{\rho(\vec{r}\,')\vec{v}\,(\vec{r}\,)d^3r'}{|\vec{r} - \vec{r}\,'|}. \tag{3b}$$

It is assumed here that both potentials go to zero far from the localized source, and in the second solution we assume $\nabla \cdot \vec{h} = 0$, the analog of a gauge choice in electrodynamics. Then both equations in (2) are Poisson equations.

It is instructive to obtain the above result from a different and physically interesting perspective, and moreover introduce parameters convenient for discussing experimental measurements [5,7]. Following Eddington we consider the Schwarzschild metric of a point mass with geometric mass $m = GM$ at a large distance r, so that m/r is small. Using isotropic spatial coordinates we expand the Schwarzschild metric as

$$\begin{aligned} ds^2 &= \frac{(1 - m/2r)^2}{(1 + m/2r)^2}\,dt^2 - (1 + m/2r)^4 d\vec{r}^{\,2} \\ &\simeq \left(1 - \frac{2m}{r} + \frac{2m^2}{r^2} + \cdots\right)dt^2 - \left(1 + \frac{2m}{r} + \frac{3m^2}{2r^2} + \cdots\right)d\vec{r}^{\,2}. \end{aligned} \tag{4}$$

Eddington (Ref. 19, p.105) suggested that this be written in terms of dimensionless parameters as

$$ds^2 \simeq \left(1 - \alpha\frac{2m}{r} + \beta\frac{2m^2}{r^2} + \cdots\right)dt^2 - \left(1 + \gamma\frac{2m}{r} + \cdots\right)d\vec{r}^{\,2}. \tag{5}$$

The Eddington parameters α and β and γ are equal to 1 for general relativity. The series (5) is a rather general form for the metric far from a spherically symmetric body. Since the constant m which appears in (5) represents the mass of the central body the parameter α may be absorbed into it, which is equivalent to taking $\alpha \equiv 1$. This is consistent so long as no independent non-gravitational determination of the mass of the central body is considered. In this work we will display α explicitly.

The parameters may be viewed as a book-keeping tool for tracking which terms in the metric contribute to some gravitational effect, for example the deflection of starlight by the sun. Alternatively they may be viewed as numbers which may not be equal to 1 if a metric theory other than general relativity is valid. In either case they provide a convenient way to express the results of experimental tests of gravity as giving values to the parameters. This parametrized approach has been extended to include many other parameters, and has been highly developed under the name parametrized post Newtonian theory, or PPN (Ref. 20; see in particular p.339). Solar system measurements give the values $\beta - 1 = (0.2 \pm 1.0) \times 10^{-3}$ and $\gamma - 1 = (-1.2 \pm 1.6) \times 10^{-3}$. In this paper we consider only general relativity and emphasize that we are not using the more general PPN approach.

We limit ourselves to phenomena in which the second order term in g_{00} is unimportant, so we may ignore β and assume that the underlying gravitational theory is linear. Then for a stationary point mass

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2}. \tag{6}$$

Since the metric is nearly Lorentz we may generalize this to a moving mass point using a transformation that is nearly Lorentz, that is to first order in the velocity

$$t_r = t - \mathrm{v}x, \qquad x_r = x - \mathrm{v}t, \tag{7}$$

where the subscript r indicates the point mass is at rest in that frame. This gives the metric for the moving mass as

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2} + (\alpha + \gamma)\frac{4m}{r}\,\mathrm{v}dx\,dt, \tag{8}$$

which obviously generalizes for motion in any direction to

$$ds^2 = \left(1 - \alpha\frac{2m}{r}\right)dt^2 - \left(1 + \gamma\frac{2m}{r}\right)d\vec{r}^{\,2} + (\alpha + \gamma)\frac{4m}{r}\,(\vec{\mathrm{v}} \cdot d\vec{r})dt. \tag{9}$$

Since we assume that the underlying theory is linear to this order we can superpose the fields of a distribution of such point masses and write for any such mass distribution

$$\frac{GM}{r} \rightarrow \phi(\vec{r}) = -G \int \frac{\rho(\vec{r}\,')d^3r'}{|\vec{r} - \vec{r}\,'|}, \tag{10a}$$

$$\frac{4GM\vec{v}}{r} \rightarrow \vec{h}(\vec{r}) = 4G \int \frac{\rho(\vec{r}\,')\vec{v}(\vec{r}\,')d^3r'}{|\vec{r} - \vec{r}\,'|}, \tag{10b}$$

and the metric is

$$ds^2 = (1 + \alpha 2\phi)dt^2 - (1 - \gamma 2\phi)d\vec{r}^{\,2} + (\alpha + \gamma)(\vec{h} \cdot d\vec{r})dt. \tag{11}$$

This agrees with the general relativity result (1) when $\alpha = \gamma = 1$ but now contains appropriate combinations of Eddington parameters. This is a very strong result in that it rests only on the Schwarzschild metric (5) which is well-verified by observation, the approximate Lorentz transformation (7), and the superposition in (10). No new parameter appears in this process. Thus a measurement of a phenomenon which depends on the cross term in the metric (11) provides a value for $\alpha + \gamma$ and does not provide a logically independent test of gravitational theory. (Some authors do not agree with this interpretation; see Ref. 21, Ch. 6.)

3. THE SCALAR POTENTIAL

We recall the derivation of a scalar potential with spherical symmetry, which generalizes easily to the oblate spheroidal case. Poisson's equation for a spherically symmetric field is

$$\nabla^2\phi(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = 4\pi G\rho(r). \tag{12}$$

Note the coordinate consistency, that both sides are functions of r only. Outside the body, where the density is zero, the equation may be directly integrated twice to obtain

$$\phi(r) = D - \frac{C}{r}. \tag{13}$$

To determine the two constants of integration, C and D, we consider very large r and compare with the Green's function solution ($3a$), which gives $C = GM$ and $D = 0$ and thus the well-known result

$$\phi(r) = -\frac{GM}{r}. \tag{14}$$

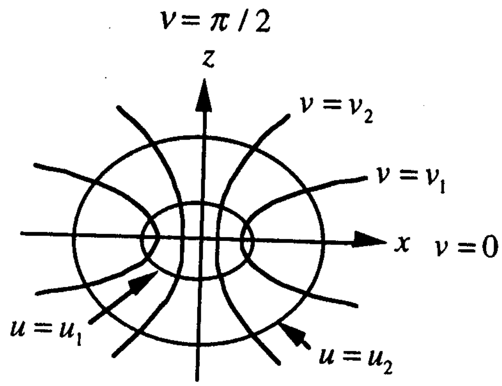

Figure 1. The oblate spheroidal coordinates: a cut through $\phi = 0$ or $\gamma = 0$. The level curves for constant u are ellipses and those for constant v are hyperbolas.

This procedure may be generalized to a potential with oblate spheroidal symmetry. To do this we use oblate spheroidal coordinates (u, v, φ) defined by [10]

$$x = a\cosh u \cos v \cos \varphi, \quad y = a\cosh u \cos v \sin \varphi, \quad z = a \sinh u \sin v. \tag{15}$$

Here u is a dimensionless radial coordinate, running from 0 to ∞, v is a latitude angle running from 0 at the equator to $\pi/2$ at the pole, and φ is the azimuth angle. The parameter a has the dimension of a distance, and we refer to it as the radial parameter (see Figure 1). At large distances, these go over to spherical coordinates, with

$$a\frac{e^u}{2} = r, \qquad v = \frac{\pi}{2} - \theta \qquad \text{(large distances).} \tag{16}$$

This form for the oblate spherical coordinates is particularly convenient for our purposes, much more so than that used by Landau and Lifshitz, for example [13]. The level surfaces for constant u are oblate ellipses of revolution; in Cartesian and spherical coordinates the equations are

$$\frac{x^2 + y^2}{a^2 \cosh^2 u} + \frac{z^2}{a^2 \sinh^2 u} = 1, \tag{17a}$$

$$\frac{r^2 \sin^2 \theta}{a^2 \cosh^2 u} + \frac{r^2 \cos^2 \theta}{a^2 \sinh^2 u} = 1. \tag{17b}$$

These equations also serve to give u as a function of the Cartesian or spherical coordinates. For a level surface the semi-major axis, semi-minor axes, flattening, and the radial parameter a are related by

$$\begin{aligned} A &= a\cosh u_o && \text{(semi-major axis)}, \\ B &= a\sinh u_o && \text{(semi-minor axis)}, \\ \frac{A-B}{A} &= 1 - \tanh u_o && \text{(flattening)}, \\ a^2 &= A^2 - B^2 && \text{(radial parameter)}. \end{aligned} \tag{18}$$

The metric of flat Euclidean space in these coordinates is

$$d\vec{r}^2 = a^2(\sinh^2 u + \sin^2 v)\,(du^2 + dv^2) + a^2\cosh^2 u \cos^2 v\, d\varphi^2. \tag{19}$$

Following our procedure for the spherical case we now seek a solution with oblate spheroidal symmetry, that is with $\phi = \phi(u)$. Note that this does not imply that the surface of the source body is an equipotential or that its density is a function of u only. Poisson's equation (2a) in oblate spheroidal coordinates is

$$\left[\frac{1}{a^2(\sinh^2 u + \sin^2 v)}\right] \frac{1}{\cosh u}\,\frac{d}{du}\left(\cosh u \frac{d\phi}{du}\right) = 4\pi G\rho. \tag{20}$$

For consistency the density must therefore depend on both u and v, with the v dependence given by

$$\rho(u,v) = \frac{P(u)}{\sinh^2 u + \sin^2 v}, \tag{21}$$

where P is a function of only u, i.e. the density is larger at the equator than at the poles. However we emphasize that this has no bearing on the external fields we calculate in this paper.

Outside the body where the density is zero the potential obeys Laplace's equation, which may be easily integrated twice,

$$\frac{d}{du}\left(\cosh u \frac{d\phi}{du}\right) = 0, \qquad \frac{d\phi}{du} = \frac{C}{\cosh u},$$

$$\phi = C\int \frac{du}{\cosh u} = D + C\arctan(\sinh u). \tag{22}$$

To evaluate the two constants of integration, C and D, we consider large u, in which case the field becomes spherically symmetric and we may relate it to the spherical case using (16). Using the large argument expansion for the arctan and comparing with (14) we find

$$\phi(u) = \frac{GM}{a}\left[\arctan(\sinh u) - \frac{\pi}{2}\right]. \tag{23}$$

We will consider the multipole expansion of this in spherical coordinates in Section 5, and its relation to the surface shape of the earth in Section 6 [11,12].

4. THE VECTOR POTENTIAL

As with the scalar potential we first do the spherical case in such a way that it generalizes easily to the oblate spheroidal case. To do this we consider eq. (2b), with $\nabla\cdot\vec{h} = 0$, and with emphasis on the axial symmetry of the field. For a rigidly rotating body the velocity field of its matter is

$$\vec{v} = \vec{\omega}\times\vec{r} = \omega(-y, x, 0). \tag{24}$$

Since the velocity curves $\vec{v}$ are circles parallel to the x, y plane we impose the same axial symmetry on the field lines $\vec{h}$ and seek solutions of the form

$$\vec{h} = f(r)\omega(-y, x, 0). \tag{25}$$

That is we assume the $\vec{h}$ field lines are also circles parallel to the x, y plane.

We now consider the first component of $\vec{h}$; observe that

$$\begin{aligned}\nabla^2(yf(r)) &= y\nabla^2 f(r) + \frac{\partial f(r)}{\partial y} = y\nabla^2 f(r) + \frac{\partial r}{\partial y}\frac{df}{dr}\\ &= y\left[\nabla^2 f(r) + \frac{2}{r}\frac{df(r)}{dr}\right].\end{aligned} \tag{26}$$

Equation (2b) for the first component of $\vec{h}$ is then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^3\frac{df}{dr}\right) + \frac{2}{r}\frac{df}{dr} = -16\pi G\rho. \tag{27}$$

Outside the body where $\rho = 0$ this is easily integrated to give the solution

$$f = D + \frac{C}{r^3}. \tag{28}$$

To determine the constants of integration, C and D, we compare this solution with the integral solution (3b) far from the source body. The constant D is obviously zero, and to find C we consider a point on the y axis very far from the central body. Then, equating the above solution and (3b), we find by expansion

$$\begin{aligned}-\frac{C}{r^3}\,\omega y = -\frac{C}{y^2}\,\omega &= -4G\omega\int\frac{\rho(r')y'd^3r'}{|r-r'|}\\ &\simeq -\frac{4G\omega}{y}\int\rho(r')y'd^3r' - \frac{4G\omega}{y^2}\int\rho(r')(y')^2d^3r'. \end{aligned}\tag{29}$$

The first integral is zero if the origin is at the center of mass, and the second integral is half the moment of inertia; we thereby obtain $C = 2GI$, and the solution

$$f = \frac{2GI}{r^3}\,, \tag{30a}$$

$$\vec{h} = \frac{2GI\omega}{r^3}\,(-y,x,0) = \frac{2GJ}{r^3}\,(-y,x,0) = \frac{2G}{r^3}\,\vec{J}\times\vec{r} \tag{30b}$$

(see Refs. 12,14, and 15).

To obtain the vector potential for oblate spheroidal symmetry we follow the above derivation closely with oblate spheroidal coordinates. The velocity of matter in the body is still given by (24), so we seek a solution of the form

$$\vec{h} = f(u)\omega(-y,x,0), \tag{31}$$

that is with field lines that are circles, the analog of the spherically symmetric solution (25). We consider the first Cartesian component of $\vec{h}$, and note that

$$\nabla^2(yf(u)) = y\nabla^2 f(u) + 2\frac{\partial f(u)}{\partial y} = y\nabla^2 f(u) + 2\frac{\partial u}{\partial y}\,\frac{df(u)}{du}\,. \tag{32}$$

A slightly tedious calculation with (17) yields

$$\frac{\partial u}{\partial y} = \frac{y}{a^2}\,\frac{\sinh u}{\cosh u(\sinh^2 u + \sin^2 v)}\,. \tag{33}$$

Substitution of (32) and (33) into (2b) then leads to

$$\frac{1}{\cosh u}\,\frac{d}{du}\left(\cosh u\frac{df}{du}\right) + 2\frac{\sinh u}{\cosh u}\,\frac{df}{du} = -16\pi G\rho a^2(\sinh^2 u + \sin^2 v). \tag{34}$$

Notice that the same consistency condition on the functional form of the density occurs here as with the scalar potential, that is (21). Outside the body, in empty space, this takes the form

$$\frac{1}{\cosh u}\left[\frac{d}{du}\left(\cosh u\frac{df}{du}\right)+2\sinh u\frac{df}{du}\right]=0. \tag{35}$$

Integration of this is straightforward. We first rewrite it as

$$\frac{1}{\cosh^3 u}\left[\frac{d}{du}\left(\cosh^3 u\frac{df}{du}\right)\right]=0, \tag{36}$$

which leads directly to

$$f=D+C\left[\arctan(\sinh u)+\frac{\sinh u}{\cosh^2 u}\right]. \tag{37}$$

To determine the constants of integration we again appeal to the large distance limit, and ask that the above agree with the spherically symmetric case (30a). This leads to

$$f=-\frac{3GI}{a^3}\left[\arctan(\sinh u)+\frac{\sinh u}{\cosh^2 u}-\frac{\pi}{2}\right]. \tag{38}$$

Thus finally we have from (31) and (38)

$$\begin{aligned}\vec{h}&=-\frac{3GI}{a^3}\left[\arctan(\sinh u)+\frac{\sinh u}{\cosh^2 u}-\frac{\pi}{2}\right]\omega(-y,x,0)\\&=-\frac{3G}{a^3}\left[\arctan(\sinh u)+\frac{\sinh u}{\cosh^2 u}-\frac{\pi}{2}\right]\vec{J}\times\vec{r},\end{aligned} \tag{39}$$

which is the exact solution for the vector potential.

5. MULTIPOLE EXPANSION

Our main results are the scalar potential (23) and the vector potential (39). These are completely determined by the total mass M and total angular momentum J of the source, and the radial parameter a. The mass and angular momentum of the earth are accurately measured; the radial parameter is simply related to the quadrupole parameter J_2 as we will show in this section, and thus a is also well-known for the earth. Therefore the scalar and vector potentials are completely determined for the earth within the context of the spheroidal symmetry assumption. We emphasize that the details of the density distribution do not matter.

For a system with small oblateness like the earth it is interesting to relate our solutions to a standard multipole expansion in spherical coordinates [7].The expansion of the scalar potential for an axially symmetric system may be written as

$$\phi(r,\theta) = -\frac{GM}{r}\left[1 - J_2\frac{R^2}{r^2}P_2(\theta) - J_3\frac{R^3}{r^3}P_3(\theta) - J_4\frac{R^4}{r^4}P_4(\theta)\cdots\right]. \quad (40)$$

Here R is any characteristic distance, and is usually taken to be the equatorial radius of the body, as we do here for the earth. To compare the oblate spheroidal solution to this we expand (23) for large distances, and express $\sinh u$ in terms of r and θ. Equation (17b) allows us to solve for $\sinh u$ as

$$\sinh^2 u = \frac{1}{2}\left\{\left(\frac{r^2}{a^2} - 1\right) + \left[\frac{r^4}{a^4} - 2\frac{r^2}{a^2}(1 - 2\cos^2\theta) + 1\right]^{1/2}\right\}. \quad (41)$$

For the potential in (23) and (41) we use the expansions

$$\arctan(\sinh u) - \frac{\pi}{2} = -\frac{1}{\sinh u} + \frac{1}{3\sinh^3 u} - \frac{1}{5\sinh^5 u}\cdots, \quad (42a)$$

$$\sinh^2 u = \frac{r^2}{a^2}\left[1 - \frac{a^2}{r^2}\sin^2\theta + \frac{a^4}{r^4}\sin^2\cos^2\theta\cdots\right]. \quad (42b)$$

Combining these in (23) we get the expansion

$$\phi = -\frac{GM}{r}\left[1 - \frac{1}{3}\frac{a^2}{r^2}P_2(\theta) + \frac{1}{5}\frac{a^4}{r^4}P_4(\theta)\cdots\right]. \quad (43)$$

By comparison with (40) we have

$$J_2 = \frac{1}{3}\frac{a^2}{r^2}, \qquad J_3 = 0, \qquad J_4 = -\frac{1}{5}\frac{a^4}{R^4}. \quad (44)$$

This allows us to express the radial parameter in terms of the measured quadrupole parameter as $a^2 = 3R^2 J_2$. An amusing aspect of the above result is a simple relation between the moments [11],

$$J_4 = -\tfrac{9}{5}J_2^2. \quad (45)$$

For the earth the equatorial radius and the moment parameters are accurately measured and given by [10,16]

$$R = 6.37814 \times 10^6\text{m}, \qquad J_2 = 1.08263 \times 10^{-3},$$
$$J_3 = -(2.4 \pm 0.3) \times 10^{-6}, \qquad J_4 = -(1.4 \pm 0.2) \times 10^{-6}. \tag{46a}$$

Thus the radial parameter is

$$a = \sqrt{3J_2}\,R = 3.63492 \times 10^5\text{m}. \tag{46b}$$

The relation (45) predicts that $J_4 = -2.1 \times 10^{-6}$, which is in reasonably good agreement with the measured value in (46a). From this and the value of J_3 we may fairly say that the oblate spheroidally symmetric potential is accurate to a few parts in a million, due of course to the smallness of the higher moment parameters.

We also wish to express the vector potential $\vec{h}$ as a Legendre type series. Since the components of $\vec{h}$ are harmonic functions we see from (25) that both $yf(u)$ and $xf(u)$ are harmonic also. Thus for the general axially symmetric case we may expand them as

$$\begin{aligned} yf(u) &= r\sin\theta\sin\varphi f(r,\theta) = N^s \sum_{\ell m} b^s_{\ell m}\frac{R^{\ell+2}}{r^{\ell+1}}\,P^m_\ell(\theta)\sin m\varphi, \\ xf(u) &= r\sin\theta\sin\varphi f(r,\theta) = N^c \sum_{\ell m} b^c_{\ell m}\frac{R^{\ell+2}}{r^{\ell+1}}\,P^m_\ell(\theta)\cos m\varphi, \end{aligned} \tag{47a}$$

$$f(r,\theta) = f(u[r,\theta]), \tag{47b}$$

where N^s and N^c are constants to be determined for convenience. For consistency the sums in (47) must be over only $m = 1$ terms. The dependence on φ is then automatically correct, and the subscripts m and the superscripts c and s are redundant and may be dropped, to give

$$\begin{aligned} f(r,\theta)\sin\theta &= N\sum_\ell b_\ell\left(\frac{R}{r}\right)^{\ell+2} P^1_\ell(\theta) \\ &= N\left[b_1\frac{R^3}{r^3}(\sin\theta) + b_2\frac{R^4}{r^4}(3\sin\theta\cos\theta)\right. \\ &\qquad \left. + b_3\frac{R^5}{r^5}\left(\frac{3}{2}\sin\theta(5\cos^2\theta - 1)\right)\cdots\right]. \end{aligned} \tag{48}$$

The common factor of $\sin\theta$ makes this a consistent expansion for f. We may determine the constant N by asking that this agree with the spherically symmetric case (30) with $b_1 = 1$, which implies that $N = 2GI/R^3$, and

$$f(r,\theta)\sin\theta = \frac{2GI}{r^3}\left[(\sin\theta) + b_2\frac{R}{r}(3\sin\theta\cos\theta) + b_3\frac{R^2}{r^2}\left(\frac{3}{2}\sin\theta(5\cos^2\theta - 1)\right)\cdots\right]. \qquad (49)$$

This is the expansion for the general axisymmetric case.

To compare with our oblate spheroidally symmetric solution (38) we expand with the help of (42) to obtain

$$f(r,\theta)\sin\theta = \frac{2GI}{r^3}\left[(\sin\theta) - \frac{1}{5}\frac{a^2}{r^2}\left(\frac{3}{2}\sin\theta(5\cos^2\theta - 1)\right)\cdots\right]. \qquad (50)$$

Comparison of this with the general expression (49) gives

$$b_1 \equiv 1, \qquad b_2 = 0, \qquad b_3 = -\frac{1}{5}\frac{a^2}{R^2}. \qquad (51a)$$

Notice that we thus have a simple relation between b_3 and the well-measured quadrupole parameter J_2,

$$b_3 = -\tfrac{3}{5}J_2. \qquad (51b)$$

In terms of the expansion we may summarize with the following expression for the vector potential:

$$\vec{h} = \frac{2G}{r^3}\left[1 - \frac{1}{5}\frac{a^2}{r^2}\frac{P_3^1(\theta)}{\sin\theta}\cdots\right]\vec{J}\times\vec{r}, \qquad (52)$$

which of course includes the spherically symmetric case as the first term.

6. ROTATION AND THE SHAPE OF THE EARTH

The preceding has been based on the assumed symmetry of the external fields and is thus independent of details of the shape and mass distribution of the source body, the earth. Here we discuss the shape of the earth and relate it to the shape of the external equipotential surfaces. If the earth was sufficiently plastic during formation we may expect its surface to be an equipotential surface in its own rotating reference frame [9]. This immediately implies that its surface cannot be an equipotential in the inertial nonrotating frame. That is the surface of the earth cannot be a surface of constant u. A derivation of the shape of the earth in the context of our solution is interesting in its own right, and moreover it relates the physical flattening of the surface to the flattening of the exterior equipotential surfaces and to the quadrupole parameter J_2, a classic problem in geodesy.

The scalar potential for a stationary body in the frame rotating with the earth at ω is given by the scalar potential in the inertial frame (23) with a centrifugal term added, that is

$$\begin{aligned}\phi_R(u,v) &= \frac{GM}{a}\left[\arctan(\sinh u) - \frac{\pi}{2}\right] - \frac{1}{2}\omega^2 r^2 \sin^2\theta \\ &= \frac{GM}{a}\left[\arctan(\sinh u) - \frac{\pi}{2}\right] - \frac{1}{2}\omega^2 a^2 \cosh^2 u \cos^2 v. \qquad (53)\end{aligned}$$

An equipotential surface in the rotating frame is described by $\phi_R =$ constant. The constant is the value of ϕ_R at the north pole, where the centrifugal term vanishes. This gives the equation for the surface as

$$\cos^2 v = \left(\frac{2}{\omega^2 a^2 \cosh^2 u}\right)\frac{GM}{a}\left[\arctan(\sinh u) - \arctan(\sinh u_p)\right]. \qquad (54)$$

It is convenient to express this in terms of the polar radius, which is $R_p = a\sinh u_p$, and a small dimensionless parameter m_c defined as the ratio of the centrifugal force at the equator to the gravitational force; then the surface equation is

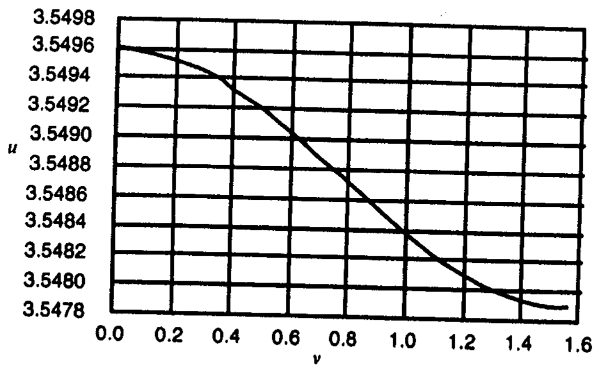

Figure 2. Surface shape of the earth according to eq. (55).

$$\cos^2 v = \frac{2}{m_c}\left(\frac{R^3}{a^3}\right)\frac{1}{\cosh^2 u}\left[\arctan(\sinh u) - \arctan\left(\frac{R_p}{a}\right)\right], \tag{55}$$
$$m_c = \frac{\omega^2 R}{(GM/R^2)}.$$

As before, R is taken to be the equatorial radius. At the equator we have $\cos v = 1$, we define $u \equiv u_e$ and from (15) $R = a\cosh u_e$; thus at the equator (55) gives

$$\begin{aligned} m_c &= 2\frac{R}{a}\left[\arctan\left(\sqrt{\frac{R^2}{a^2}-1}\right) - \arctan\left(\frac{R_p}{a}\right)\right] \\ &= 2\frac{R}{a}\left[\arctan\left(\sqrt{\frac{R^2}{a^2}-1}\right) - \arctan\left(\frac{R_p}{R}\frac{R}{a}\right)\right]. \end{aligned} \tag{56}$$

The way in which we will actually use the shape equations (55) and (56) is to take the accurately measured values of the equatorial and polar radii, R and R_p, as known, and solve (56) numerically for the ratio R/a. This ratio yields a number for the quadrupole parameter J_2 from (44), which we may consider a theoretical prediction to be compared with the measured value in (46). For the radii and the centrifugal parameter we use [10,16]

$$R = 6.37814\times10^6\text{m}, \quad R_p = 6.35666\times10^6\text{m}, \quad m_c = 3.44252\times10^{-3}, \tag{57}$$

and find

$$\frac{R}{a} = 17.4145, \qquad J_2 = \frac{1}{3}\frac{a^2}{R^2} = 1.09915\times10^{-3}. \tag{58}$$

This is in reasonably good agreement with the measured value in (46), about 1.5% larger. With the parameters in the shape equation (55) all consistently determined we may use it to plot u as a function of v, which is shown in Figure 2.

There is an interesting and well-known approximate relation between the centrifugal parameter m_c, the flattening of the earth, and the quadrupole parameter J_2 [9] which we may obtain from the shape equation (56). We expand (56) to second order in the radial parameter a to find

$$m_c = 2\left[\frac{R-R_p}{R_p} + \frac{1}{3}\left(\frac{a^2}{R^2} - \frac{a^2R_p}{R^3}\right) - \frac{a^2}{2R^2}\right]. \tag{59}$$

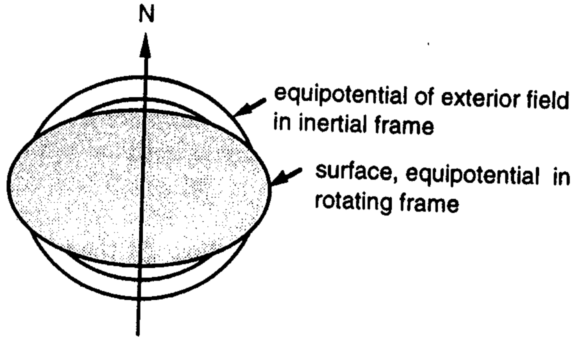

Figure 3. Relation of the equipotentials of the exterior field to the surface of the earth.

In terms of the flattening of the surface, defined as $f \equiv (R - R_p)/R$, this gives to lowest order

$$m_c = 2f - \frac{a^2}{R^2} = 2f - 3J_2\,. \tag{60}$$

Viewed as a prediction for the quadrupole parameter this yields the well-known relation

$$J_2 = \frac{2}{3}f - \frac{m_c}{3} = 1.09766 \times 10^{-3} \tag{61}$$

which is slightly more accurate than obtained above in (58).

Another way to view this result is as a relation between the flattening of the earth and the flattening of the external equipotentials near the surface. The flattening of the external equipotentials, $f_{\rm eq}$, near the earth's surface is easily obtained from (18) as

$$a^2 = A^2 - B^2 = \left(\frac{A-B}{A}\right)A(A+B) \approx f_{\rm eq}2R^2, \qquad f_{\rm eq} \approx \frac{a^2}{2R^2}, \tag{62}$$

so (60) tells us that

$$f = f_{\rm eq} + \frac{m_c}{2}\,. \tag{63}$$

Numerically the surface flattening is roughly twice the flattening of the equipotentials. The relation is shown schematically in Figure 3.

The above analysis provides a consistency check of the symmetry of the external field and the shape of the earth's surface. The value of the quadrupole parameter is of course directly measured, and we do not need this analysis to determine it. The analogous parameters for the vector field are at present not measured, but as noted previously are in principle measurable by the precession of the gyroscopes in the Gravity Probe B satellite. Due to the symmetry assumption our analysis thus gives the vector potential with no unmeasured parameters.

7. SUMMARY AND CONCLUSIONS

With the assumption of oblate spheroidal symmetry for the earth's exterior scalar and vector potentials we have obtained exact solutions in linearized general relativity theory. These solutions do not depend on details of the internal mass distribution but only on the total mass and angular momentum, and of course on the radial parameter a. Since the radial parameter may be obtained from the quadrupole parameter all of the parameters in the solutions are accurately known.

The consistency and accuracy of the assumption of oblate spheroidal symmetry may be inferred from the relation of the moment parameters J_2 and J_4 in (44) and by the small value of the measured J_3 in (45). As noted following (46b) the scalar field is accurate to about 10^{-6}. The relation between J_2 and the flattening of the earth's surface in (58) is consistent with oblate spheroidal symmetry to about a percent. Since the correction to the spherically symmetric part of the vector potential is about 10^{-3} we may expect it to be accurate to about 10^{-5}.

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Note added in proof: We have learned of related work by P. Teyssandier, in which the metric functions are obtained using specific axisymmetric models of the Earth [22]. Our results are in reasonably good agreement. Teyssandier also applies his results to the Lense–Thirring precession to be measured by Gravity Probe B; the effect of oblateness on that precession is not large enough to be observed with the presently expected accuracy.

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