Polhode Motion, Trapped Flux, and the GP-B Science Data Analysis

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and the Polhode/Trapped Flux Mapping Task Team

Outline

1. **Gyro Polhode Motion, Trapped Flux, and GP-B Readout (4 charts)**
2. **Changing Polhode Period and Path: Energy Dissipation (4 charts)**
3. **Trapped Flux Mapping (TFM): Concept, Products, Importance (7 charts)**
4. **TFM: How It Is Done - 3 Levels of Analysis (11 charts)**
   - A. **Polhode phase & angle**
   - B. **Spin phase**
   - C. **Magnetic potential**
5. **TFM: Results (9 charts)**
6. **Conclusion. Future Work (1 chart)**

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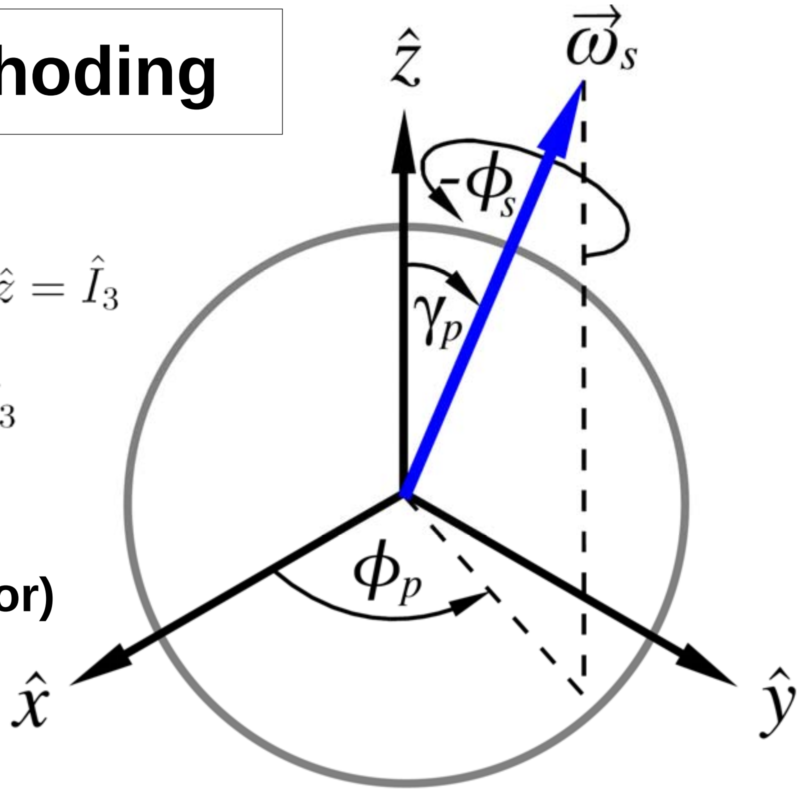
1.1 Free Gyro Motion: Polhoding



- **Euler motion equations**
  - **In body-fixed frame:** $\hat{x} = \hat{I}_1, \quad \hat{y} = \hat{I}_2, \quad \hat{z} = \hat{I}_3$
  - **With moments of inertia:**

$$0 < I_1 \leq I_2 \leq I_3$$

  - **Asymmetry parameter:**

$$0 \leq Q^2 = \frac{I_2 - I_1}{I_3 - I_1} \leq 1$$

(*Q=0* – symmetric rotor)

- **Euler solution: instant rotation axis precesses about rotor principal axis along the polhode path (angular velocity Ω_p)**

- **For GP-B gyros**

$$\frac{I_i - I_j}{I_k} \sim 10^{-6} \;\Rightarrow\; \Omega_p \sim 10^{-6}\,\omega_s$$

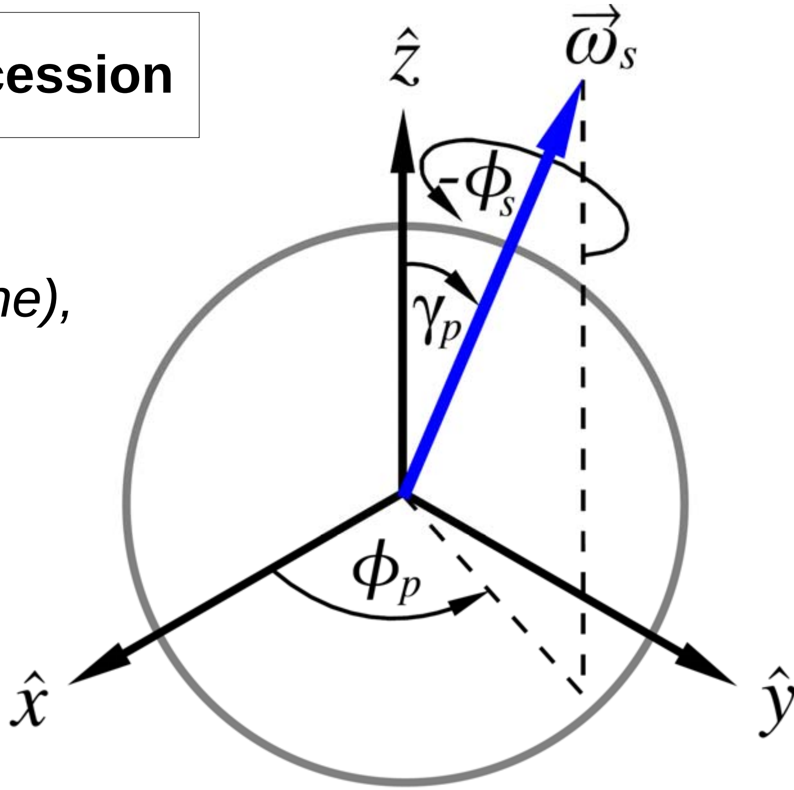
1.2 Symmetric vs. Asymmetric Gyro Precession

- **Symmetric** ($I_1 = I_2$, $Q=0$):

γ_p= const (ω_3=const, polhode path=circular cone),

$\dot{\phi}_p = \Omega_p = const$, motion is uniform,

$\phi_p(t)$ is linear function of time



- **Asymmetric** ($I_1 \neq I_2$, $Q>0$):

$$\gamma_p \neq const\,(\omega_3 \neq const,\ polhode\ \ path\ \ is$$

$$not\ \ circular),\ \dot{\phi}_p \neq const,\ \phi_p(t)\ \ is\ \ non\,linear,$$

$$motion\ \ is\ \ non-uniform$$

Why is polhoding important for GP-B data analysis? Main reason: SQUID Scale Factor Variations due to Trapped Flux

1.3 GP-B Readout: London Moment & Trapped Flux

- **SQUID signal ~ magnetic flux through pick-up loop (rolls with the S/C):**
  - **from dipole field of London Moment (LM) aligned with spin**
  - **from multi-pole Trapped Field (point sources on gyro surface – fluxons)**
- **LM flux** $\Phi^{LM}(t) = C_g^{LM}\beta(t)$ **- angle between LM and pick-up loop (** β*~10^-4* **, carries relativity signal at low roll frequency** *~ 0.01 Hz***)**
- **Fluxons**
  - **frozen in rotor surface spin, with it; transfer function 'fluxon position – pick-up loop flux' strongly nonlinear** ⟶
  - **Trapped Flux (TF) signal contains multiple harmonics of spin; spin axis moves in the body (polhoding)** ⟶
  - **amplitudes of spin harmonics are modulated by polhode frequency** ⟶

$$\Phi^{TF}(t) = \sum_n H_n(t)e^{-in(\phi_s \pm \phi_r)} = \sum_{n=odd} H_n(t)e^{-in(\phi_s \pm \phi_r)} + \beta(t)\sum_{n=even} h_n(t)e^{-in(\phi_s \pm \phi_r)}$$

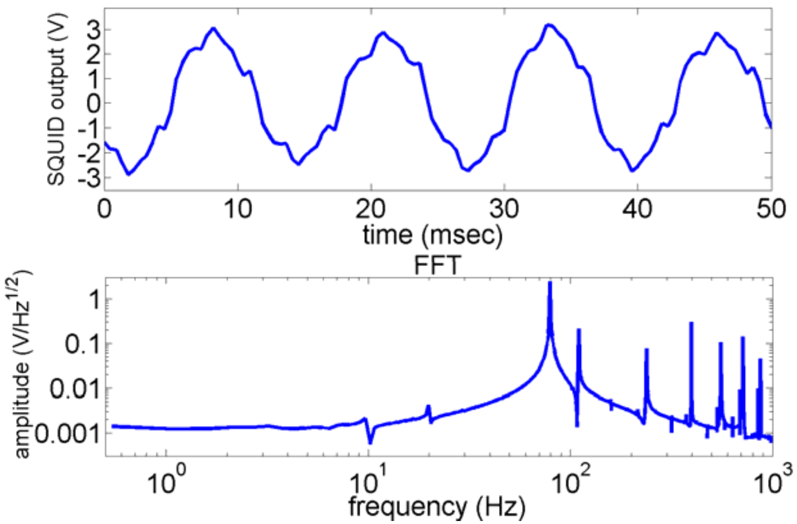
- **LM flux and LF part of Trapped Flux** *(n=0)* **combine to provide LOW FREQUENCY SCIENCE READOUT (TF** ≤ 0.05 **LM Flux):**

$$\Phi_{LF}(t) = \Phi^{LM}(t) + \Phi_{DC}^{TF}(t) = C_g^{LM}\beta(t) + C_g^{TF}(t)\beta(t); \quad C_g^{TF}(t) \equiv h_0(t)$$

1.4 GP-B High Frequency Data

- **HF SQUID Signals**
  - **FFT of first 6 spin harmonics**
  - **'snapshot': ~ 2 sec of SQUID signal sampled at 2200 Hz**
- **Both available during GSI only; ~1 snapshot in 40 sec; up to 2 day gaps in snapshot series**
- **FFT analyzed during the mission**
- **976,478 snapshots processed after the mission [harmonics $H_n(t)$]**

Gyro 1 snapshot, 10 Nov. 2004



- **LF SQUID signal (taken after additional 4 Hz LP filter) is used for relativistic drift determination ('science signal')**

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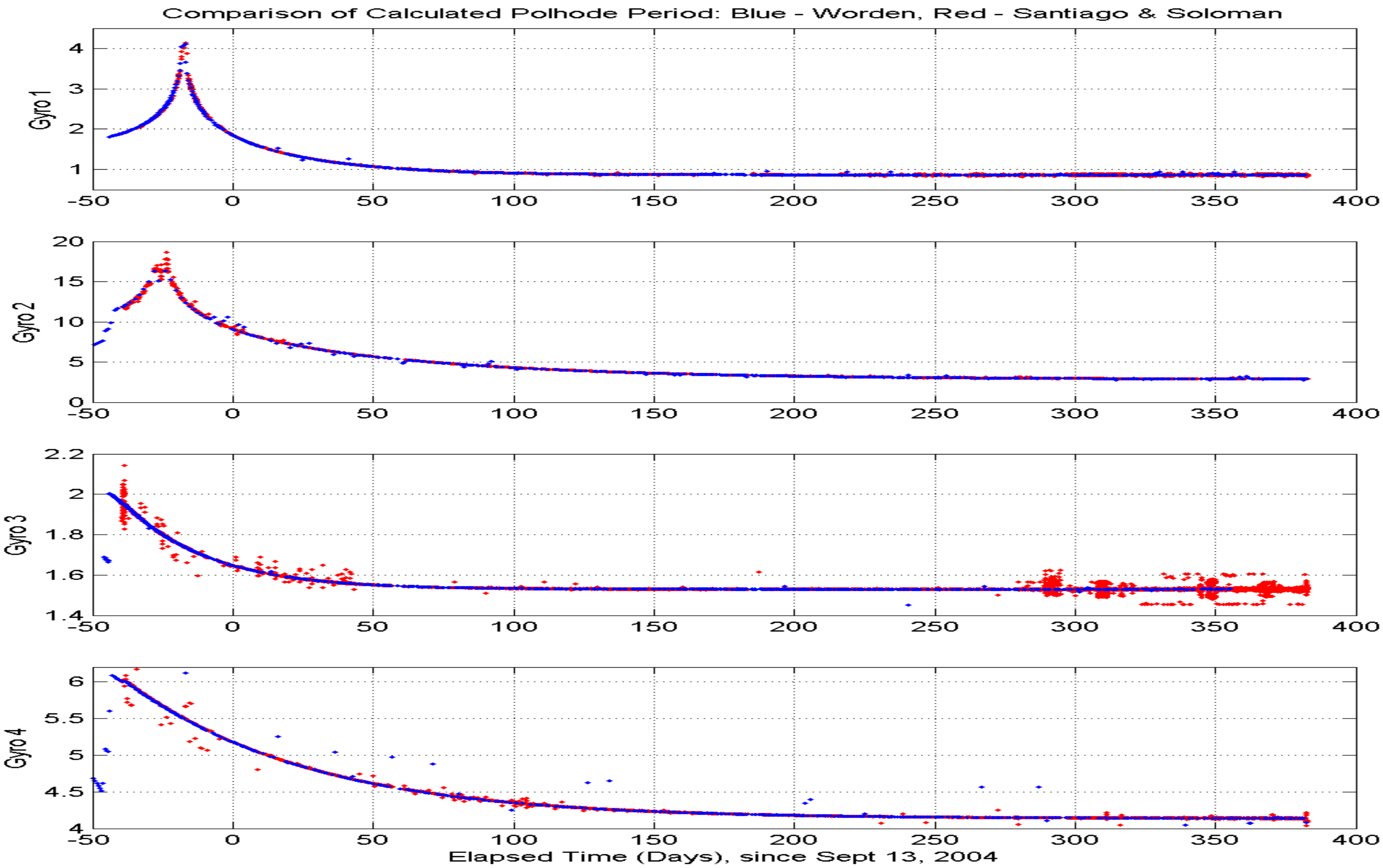
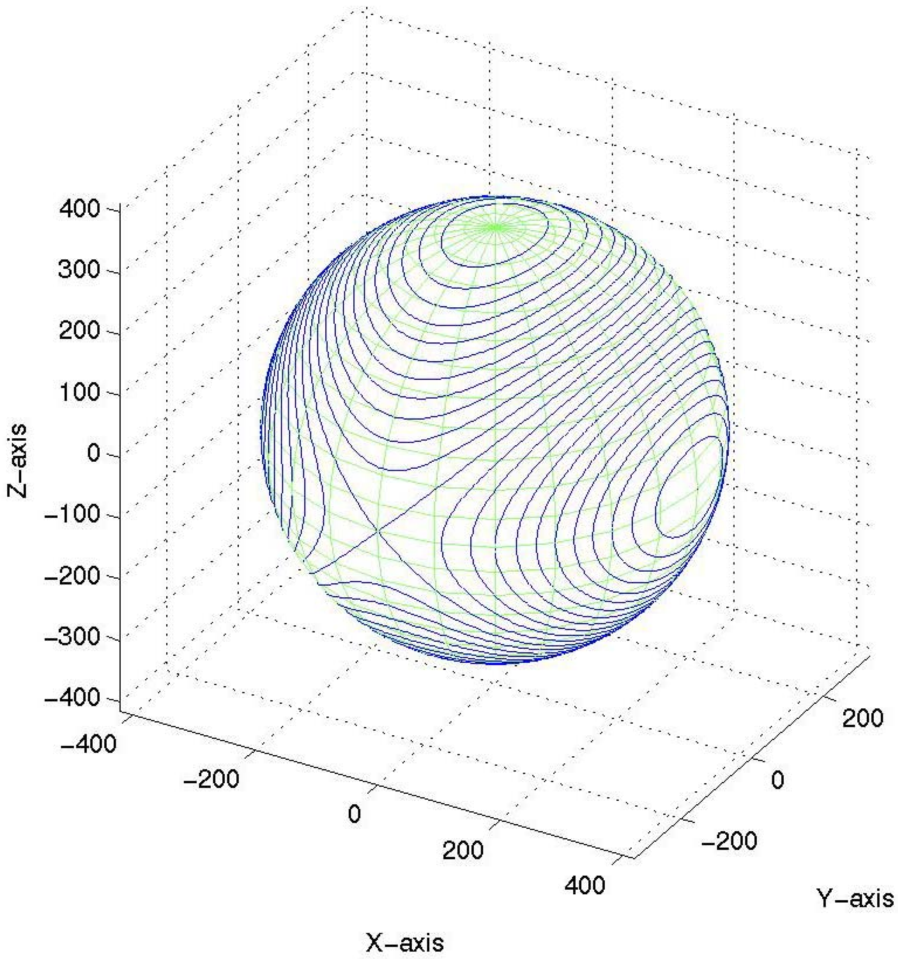
2.1 Discovery: Changing Polhode Period- from Two Sources (HF FFT- red, SRE snapshots - blue)



Also confirmed by the analysis of gyro position signal

2.2 Explanation of Changing Polhode Period: Kinetic Energy Dissipation

$(I_2 - I_1)/(I_3 - I_1) = Q^2 = 0.5$



$$L^2 = I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2$$
$$2E = I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2$$

- Classical polhode paths (blue) for given angular momentum and various energies: intersection of ellipsoids L^2 = const and E = const (no dissipation)
- Dissipation: L conserved, but E goes down *slowly*, then…
- The system slips from a curve to the nearby one with a lower energy (each path corresponds to some energy value). So the long-term path projected on {x–y} plane becomes a tight in-spiral, instead of an ellipse.

2.3 Explanation (contd.): Kinetic Energy Dissipation

- **Dissipation moves spin axis in the body to the maximum inertia axis I_3 where energy is minimum, under conserved angular momentum constraint**
- **Relative total energy loss from min, I_1, to max, I_3, inertia axis is:**

$$L = I_1\omega_1 = I_3\omega_3 \Rightarrow (E_1 - E_3)/E_1 = (I_3 - I_1)/I_3 \leq 4\times10^{-6}$$ **for GP-B gyros!**

- **The total energy loss in GP-B gyros needed to move spin axis all the way from min to max inertia axis is thus less than** *4 μJ (E ~ 1 J);* **in one year, the average dissipation power need for this is just** 10^{-13} *W* **!**
- **General dissipation model is found in the form of an additional term in the Euler motion equations (unique up to a scalar factor).**
- **Fitting the model polhode period time history to the measured one allowed the determination the rotor asymmetry parameter** Q^2 **(also from gyro position signal), the asymptotic polhode period** T_{pa} *~ 1-2 hr*, **and the characteristic time of dissipation** τ_{dis} *~ 1-2 months* **(for each gyro)**

2.4 Dissipation Modeling: Products

1. **Asymptotic Polhode Period and Dissipation Time**

	Gyro 1	Gyro 2	Gyro 3	Gyro 4
T_{pa} (hrs)	0.867	2.581	1.529	4.137
T_p (hrs) (9/4/2004)	2.14	9.64	1.96	5. 90
τ_{dis} (days)	31.9	74.6	30.7	61.2

Dissipation is slow ($T_p << \tau_{dis}$),

so the polhode motion of GP-B gyros is quasi-adiabatic

2. **Polhode phase and angle for the whole mission for each gyro (not perfectly accurate, but enough to start science analysis and TFM)**

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3.1 Trapped Flux Mapping (TFM): Concept

- **Trapped Flux Mapping: finding distribution of trapped magnetic field and characteristics of gyro motion from *odd* spin harmonics of HF SQUID signal by fitting to their theoretical model**
- **Scalar magnetic potential in the body-fixed frame is**

$$\Psi^{TF}(r,\theta,\phi) = \frac{\Phi_0}{2\,r_g}\sum_{l=1}^{\infty}\left(\frac{r_g}{r}\right)^{l+1}\frac{1}{l+1}\sum_{m=-l}^{l} A_{lm}\,Y_{lm}(\theta,\phi)$$

$$A_{lm} = \sum_{k=1}^{K}\left[Y_{lm}(\theta_k^+,\phi_k^+) - Y_{lm}^*(\theta_k^-,\phi_k^-)\right]$$

- **If fluxon number and positions were known, then coefficients A_{lm} are found uniquely by this formula; in reality, coefficients A_{lm} to be estimated by TFM**

3.2.TFM Concept: Key Points

- **HF SQUID signal and its preparation for TFM**

measured → $$z^{HF}(t) = \sum_{n=-\infty}^{\infty} H_n(t)\, e^{-in(\phi_s \pm \phi_r)}$$

- **TFM is *linear* fit of A_{lm} coefficients to *odd* spin harmonics using their theoretical expressions**

$$H_n(t) = \frac{\Phi_0}{2} \sum_{\substack{l=|n| \\ l \text{ odd}}}^{\infty} \left(\frac{r_g}{b}\right)^l \sum_{m=-l}^{l} A_{lm}\, d^l_{n0}\left(\frac{\pi}{2}\right) d^l_{mn}(\gamma_p)\, e^{im\phi_p} I_l$$ **, n odd**

- **Knowing A_{lm}, ϕ_p & γ_p, can predict scale factor due to TF**

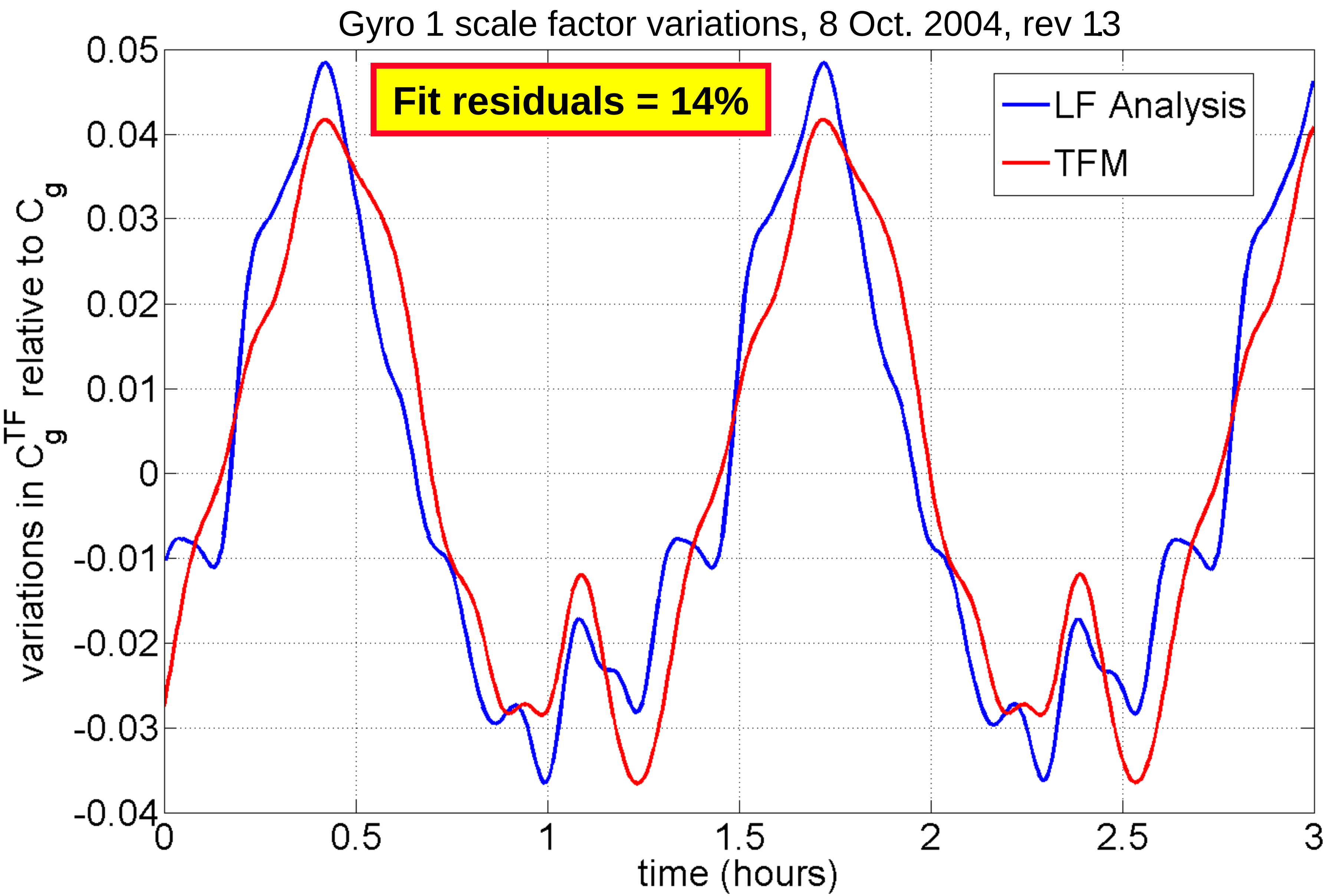
$$C_g^{TF}(t) = \left.\frac{\partial H_0(t)}{\partial \beta}\right|_{\beta=0} = \frac{\Phi_0}{2} \sum_{\substack{l=|n| \\ l \text{ odd}}}^{\infty} \left(\frac{r_g}{b}\right)^l \sum_{m=-l}^{l} A_{lm}\, l\, P_{l-1}(0)\, d^l_{m0}(\gamma_p)\, e^{im\phi_p} I_l$$

measured data **nonlinear parameters** **linear parameters**

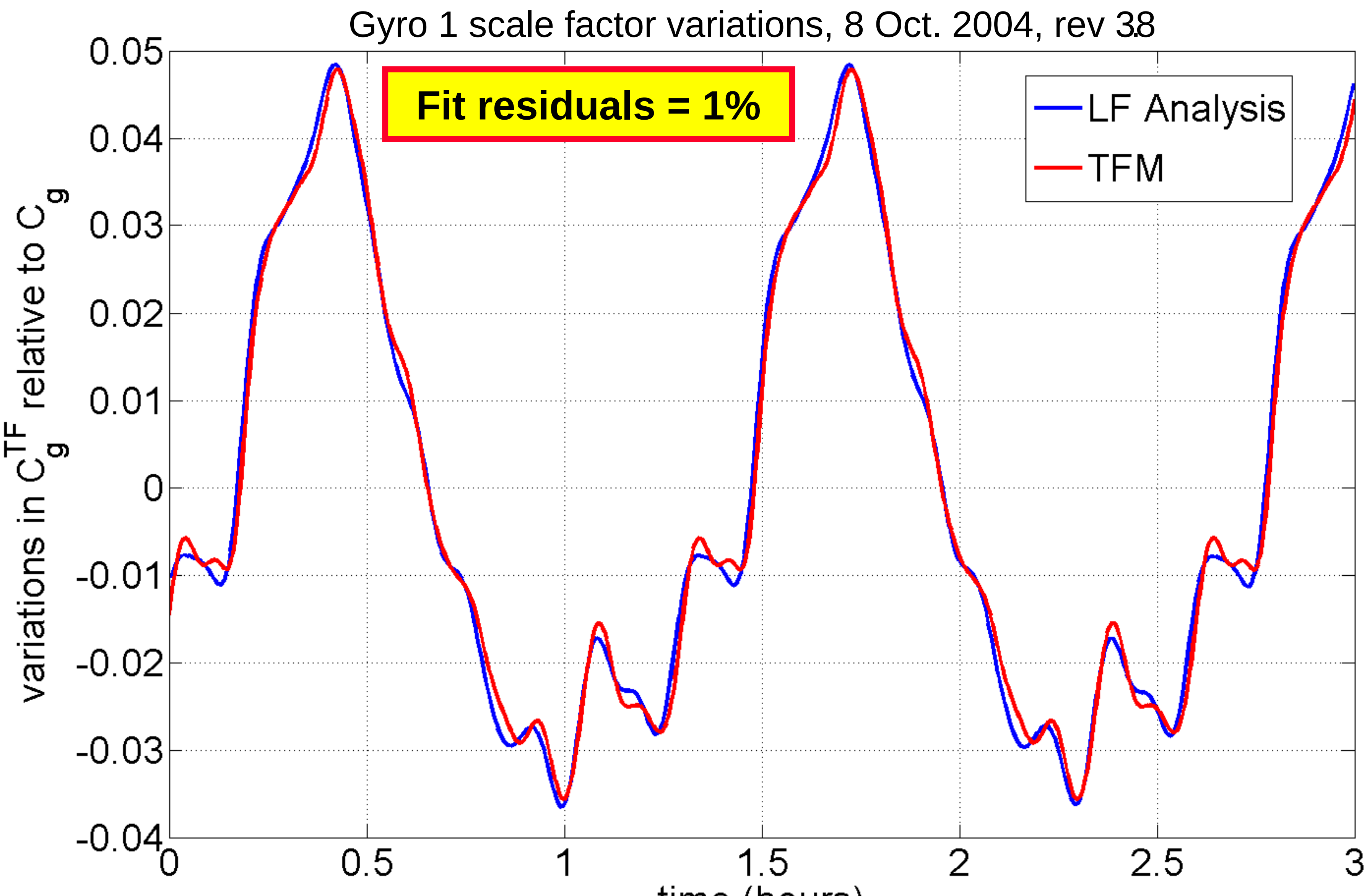
3.3 TFM: Products

- **For each gyro/entire mission, TFM provides:**
  - **Rotor spin speed to ~ 10 nHz**
  - **Rotor spin down rate to ~ 1 pHz/s**
  - **Rotor spin phase to ~ 0.05 rad**
  - **Rotor asymmetry parameter Q^2**
  - **Polhode phase to ~ 0.02 rad (1^0)**
  - **Polhode angle to ~ 0.01 – 0.1 rad**
  - **Polhode variations of SQUID scale factor [i.e., Trapped Flux scale factor, $C_g^{TF}(t)$]**

3.4 Scale Factor Variations (Nov. 2007)



3.5 Scale Factor Variations (Aug. 2008)



3.6 TFM: Importance – Scale Factor & Torque

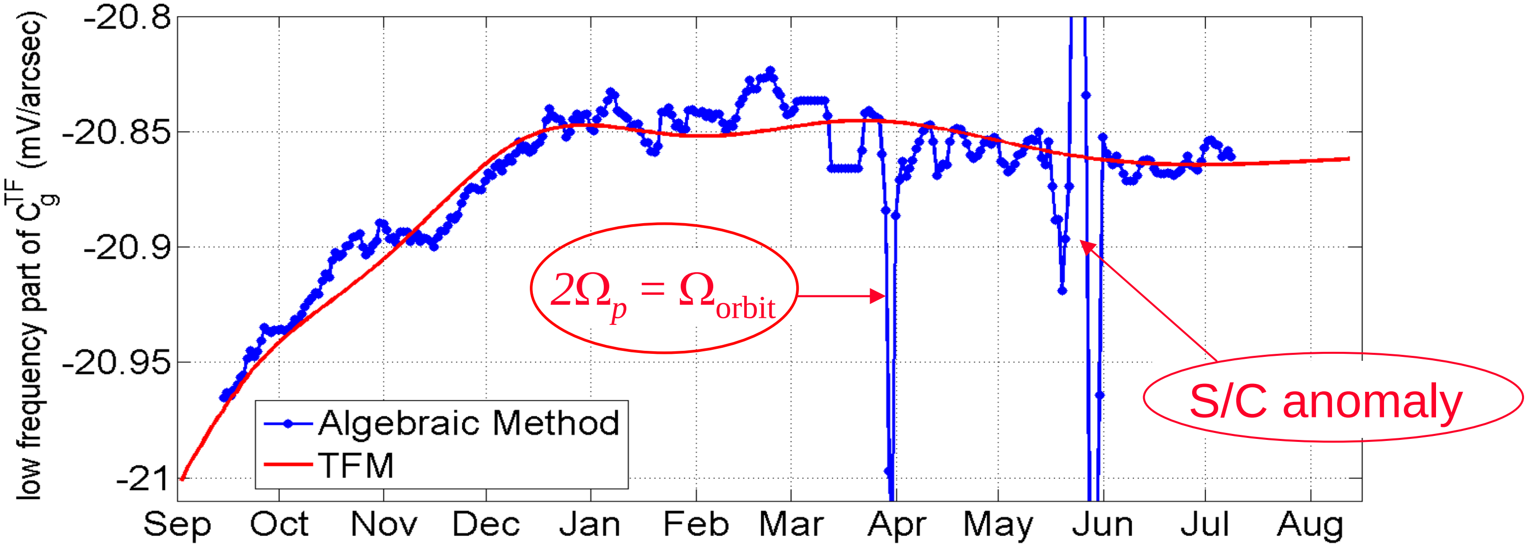
- **LF science signal analysis cannot be done w/o accurate polhode phase and angle from TFM (determination of scale factor polhode variations)**
- **Patch effect torque modeling also cannot be done w/o accurate polhode phase and angle from TFM (all the torque coefficients are modulated by polhode frequency harmonics, same as the scale factor is)**
- **TFM produces those polhode variations of scale factor from HF SQUID data (independent of LF science analysis)**
  - **Allows for separate determination of the London Moment scale factor and D.C. part of Trapped Flux scale factor slowly varying due to energy dissipation (next slide)**
  - **When used in LF science analysis, simplifies it significantly (dramatically reduces the number of estimated parameter, makes the fit linear)**

3.7 TFM Importance: D.C. Part of Scale Factor

- **SQUID Scale Factor,** $C_g(t) = C_g^{LM} + C_g^{TF}(t)$

  $C_g^{TF}(t)$ **contains polhode harmonics & D.C. part**

D.C. Part of Gyro 2 Scale Factor



With $C_g^{TF}(t)$ known through the mission, C_g^{LM} can be determined to ~ *3×10^{-5}*

GP-B Polhode/TFM Task Team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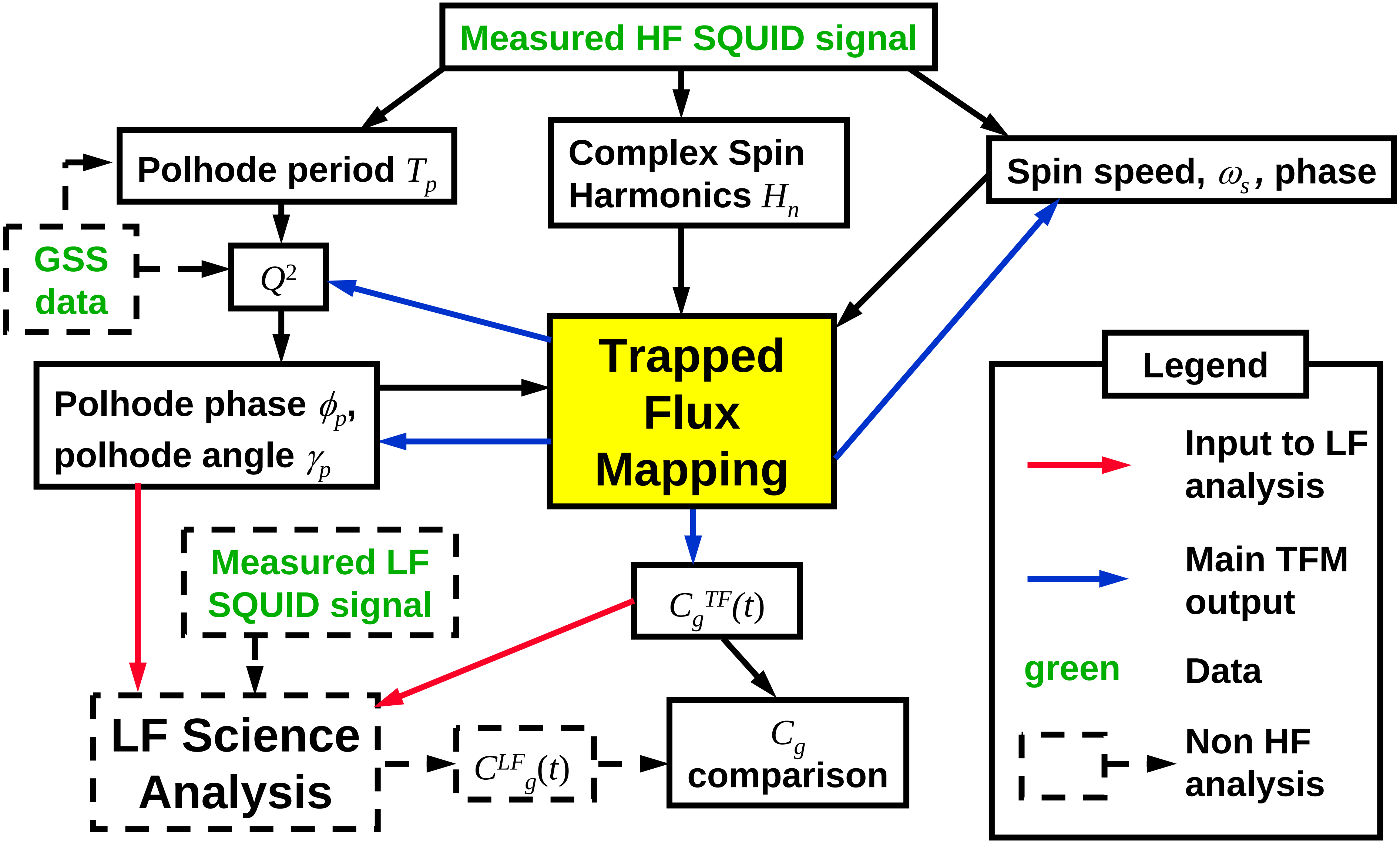
4.1 TFM & Scale Factor C_g Modeling Overview



4.2 TFM Methodology

- **Expand scalar magnetic potential in spherical harmonics**
- **Fit theoretical model to odd harmonics of spin, accounting for polhode & spin phase**
- **3 Level approach**
  - **Level A – Independent day-to-day fits, determine best polhode phase ϕ_p & angle γ_p (nonlinear)**
  - **Level B – Consistent best fit polhode phase & angle, independent day-to-day fits for spin phase ϕ_s (nonlinear)**
  - **Level C – With best fit polhode phase, angle & spin phase, fit single set of A_{lm}s to long stretches of data (linear)**
    - **» Compare spin harmonics to fit over year, refine polhode phase**

Iterative refinement of polhode phase & A_{lm}s

4.3 Level A: Polhode Phase ϕ_p & Angle γ_p

- **Level A input:**
  - **Measured spin harmonics H_n from HF SQUID signal (*n* odd)**
  - **Measured polhode frequency**
  - **Measured spin speed**
- **Fit 1-day batch ⇒ initial polhode phase for each batch**
- **Build 'piecewise' polhode phase for the entire mission, accounting for 2π ambiguities**
- **Fit exponential model to polhode phase & compute angle**

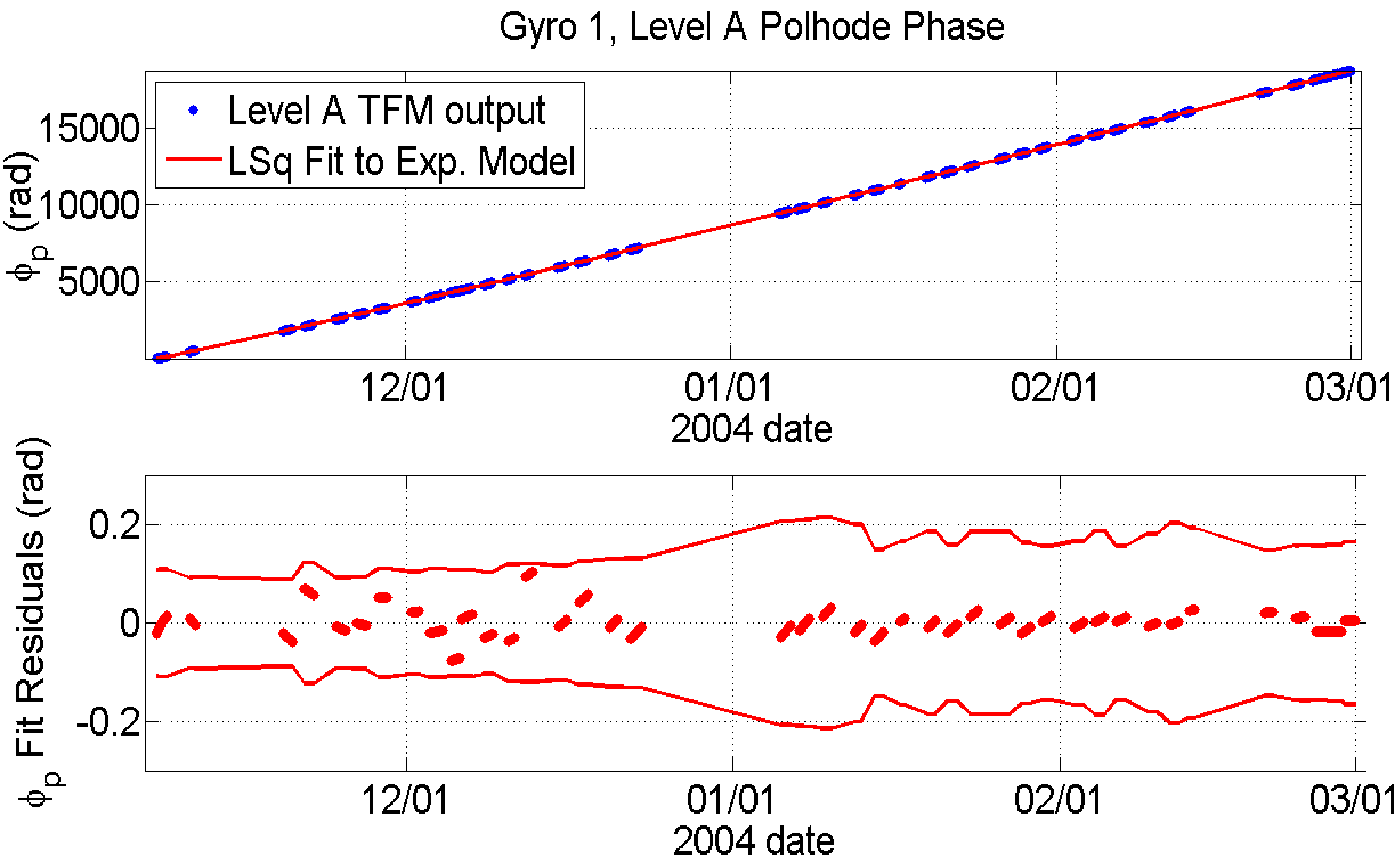
from dissipation model

$$\phi_p(t) = \phi_{p0} + \Omega_{pa}(t - t_{\text{ref}}) - \sum_{m=1}^{M} D_m \frac{\tau_{\text{dis}}}{m} e^{-m\frac{t-t_{\text{ref}}}{\tau_{\text{dis}}}} + \Delta\phi_p(t, Q^2)$$

zero when $Q^2=0$

- **Level A output:**
  - **consistent polhode phase & angle for entire mission**

4.4 Polhode Phase Determination, Level A



RMS of residuals ~ 0.1 rad (6º), or 1 part in 10^5

4.5 Level B: Spin Phase ϕ_s Estimation

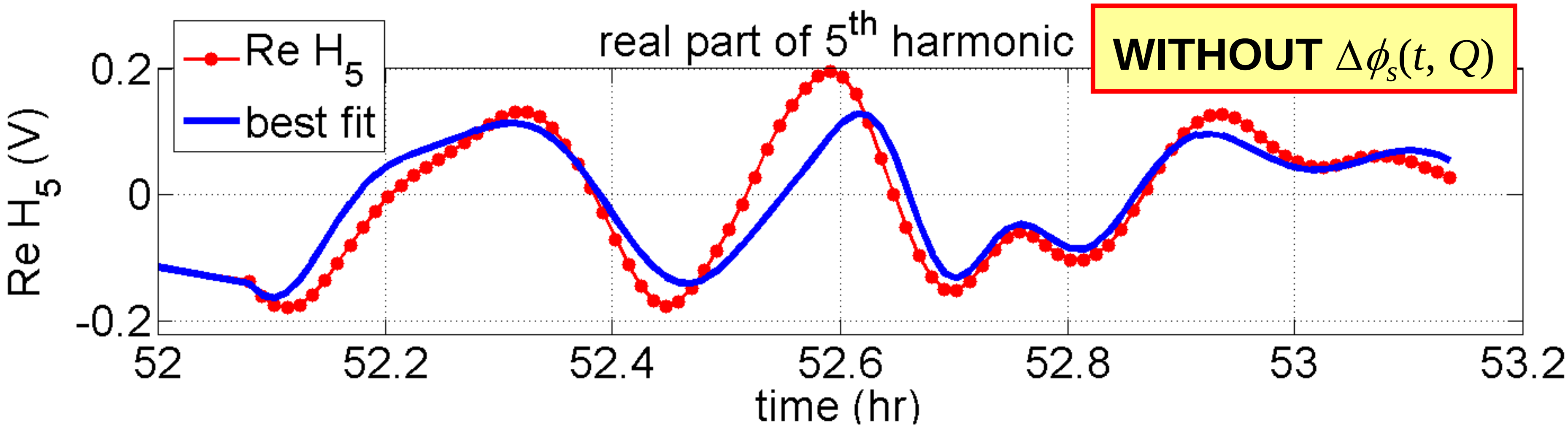
- **Level B input:**
  - **Best-fit, consistent polhode phase & angle from Level A**
  - **Measured spin harmonics H_n *(n odd)* from HF SQUID signal**
  - **Measured spin speed**
- **Fit quadratic model for spin phase, once per batch**

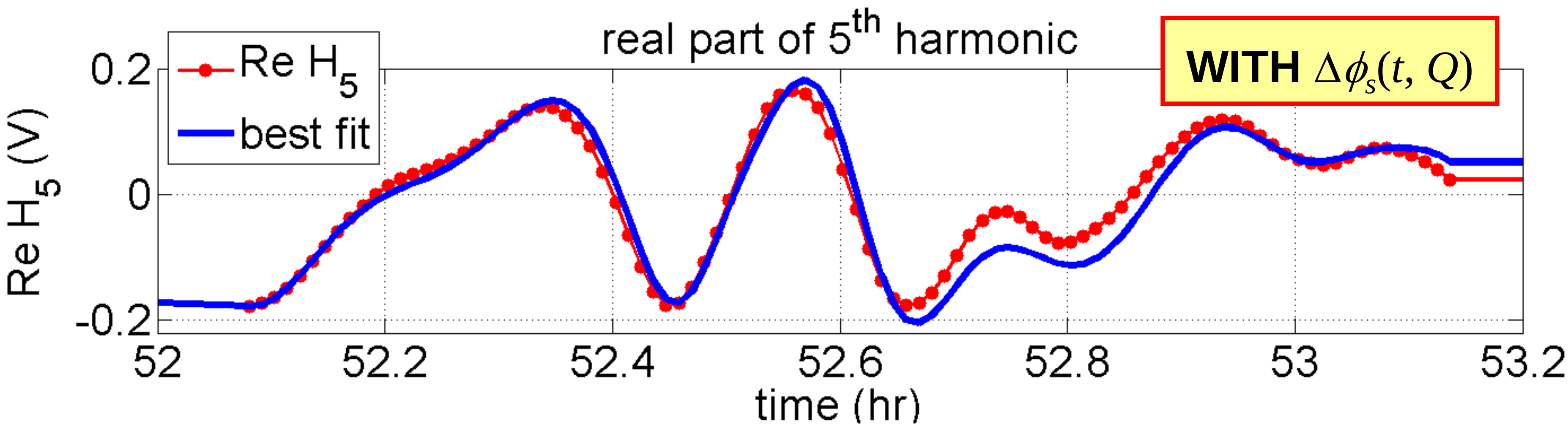
$$\phi_s(t) = \phi_{si} + \omega_i(t - t_i) + \frac{1}{2}d\omega(t - t_i)^2 + \phi_p(t, Q^2 = 0) + \Delta\phi_s(t, Q^2)$$

polhode phase
w/o asymmetry correction

- **Level B output:**
  - **Rotor spin speed to ~ 10 nHz**
  - **Rotor spin-down rate to ~ 1 pHz/s**
  - **Rotor spin phase to ~ 0.05 rad (3°)**

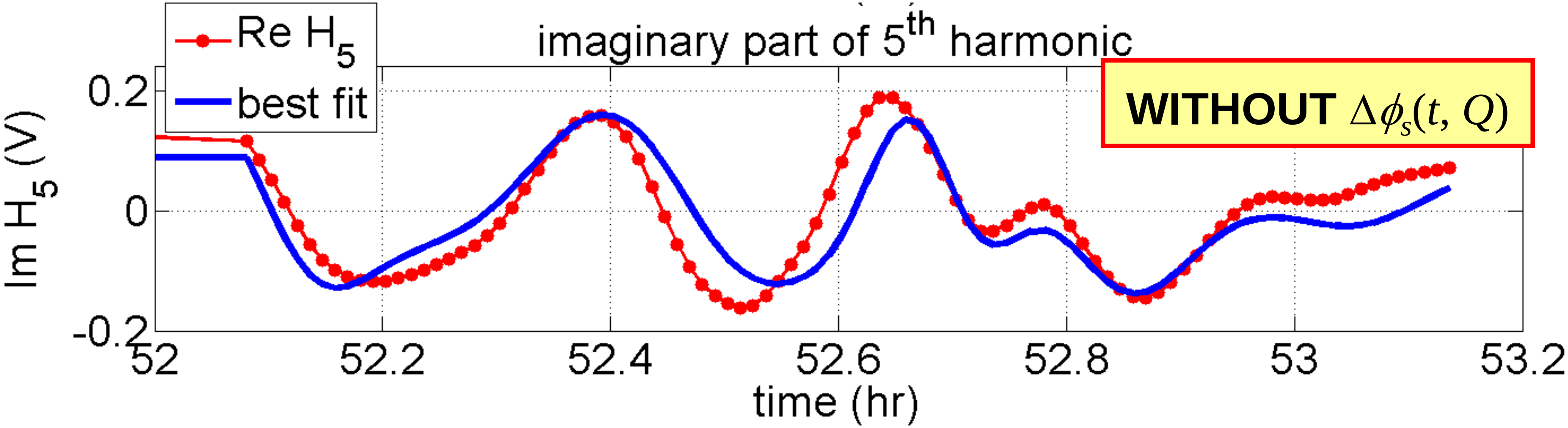
4.6 Gyro 1 Fit to H_5 with & without Extra Term



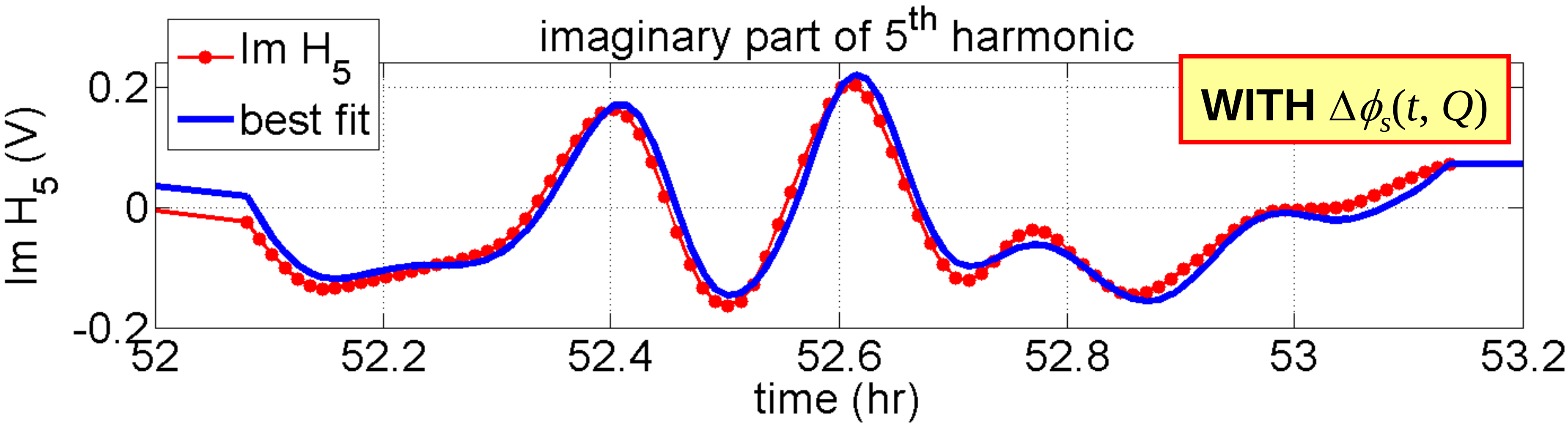


Post-fit residuals reduced by factor of 2-4

4.7 Gyro 1 Fit to H_5 with & without Extra Term



WITHOUT $\Delta\phi_s(t, Q)$



WITH $\Delta\phi_s(t, Q)$

Post-fit residuals reduced by factor of 2-4

4.8 Level C: A_{lm} & Polhode Phase Refinement

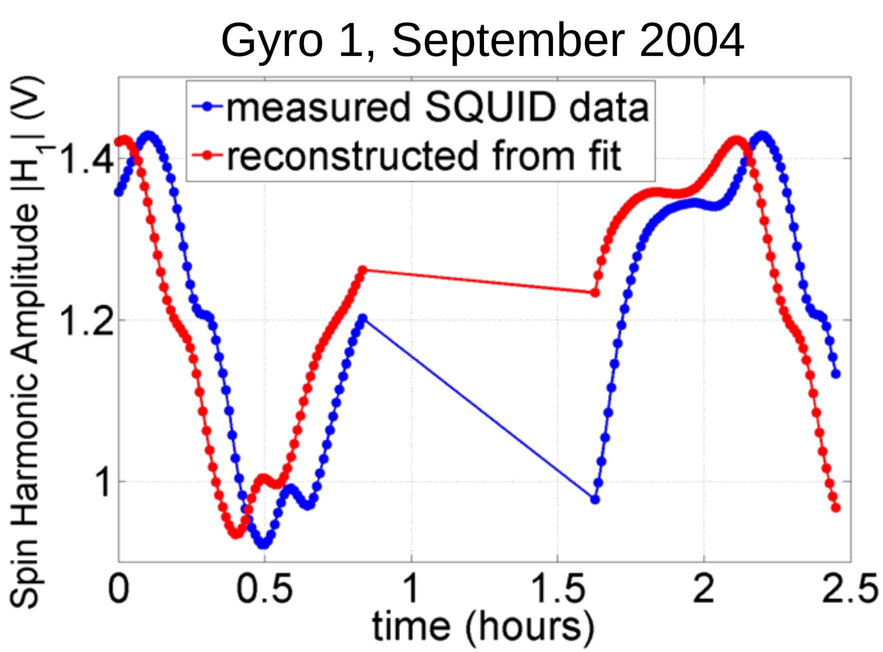
- **Level C input:**
  - **Best-fit, consistent polhode phase & angle, Q^2 - from Level A**
  - **Spin phase from Level B**
  - **Measured spin harmonics H_n from HF SQUID signal (*n* odd)**
- **Linear LSQ fit over entire mission $\Rightarrow A_{lm}$'s**
- **Level C output:**
  - **Coefficients of magnetic potential expansion, A_{lm}**
  - **Refined polhode phase & angle**

- **Polhode phase refinement**
  - **Complex H_n, accounting for elapsed spin phase, required for *linear* fit**
  - **Amplitude of spin harmonics $|H_1|$ unaffected by spin phase errors**

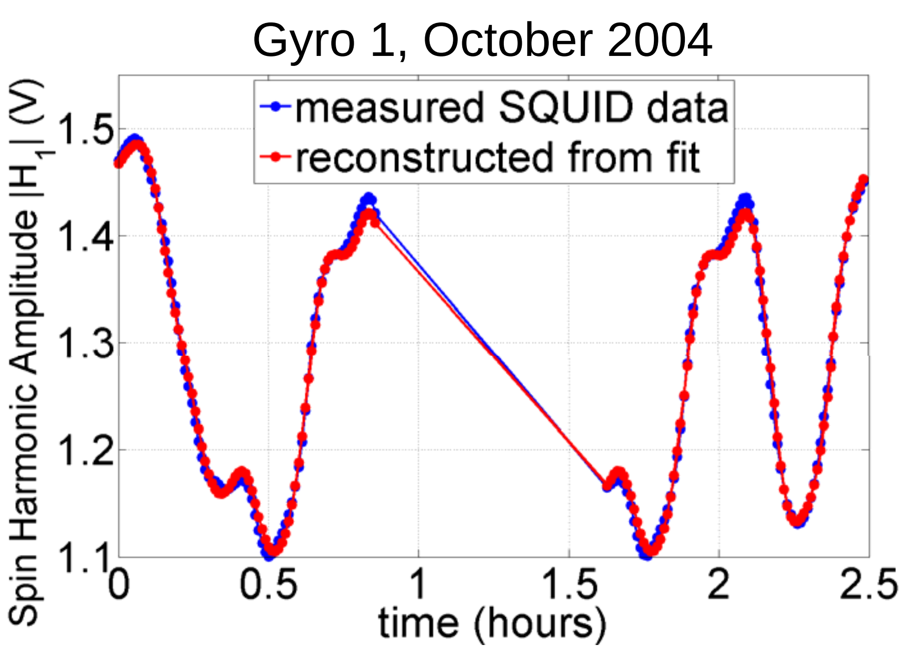
    $\Rightarrow |H_1|$ most reliable, only contains A_{lm}'s & polhode phase ϕ_p
  - **Assume A_{lm}'s correct, adjust polhode phase to match data & iterate**

4.9 Polhode Phase Refinement (Level C)

1. **Compare amplitude of spin harmonic $|H_1|$ to reconstructed version from best-fit parameters**
2. **Adjust polhode phase to match**



Phase slip

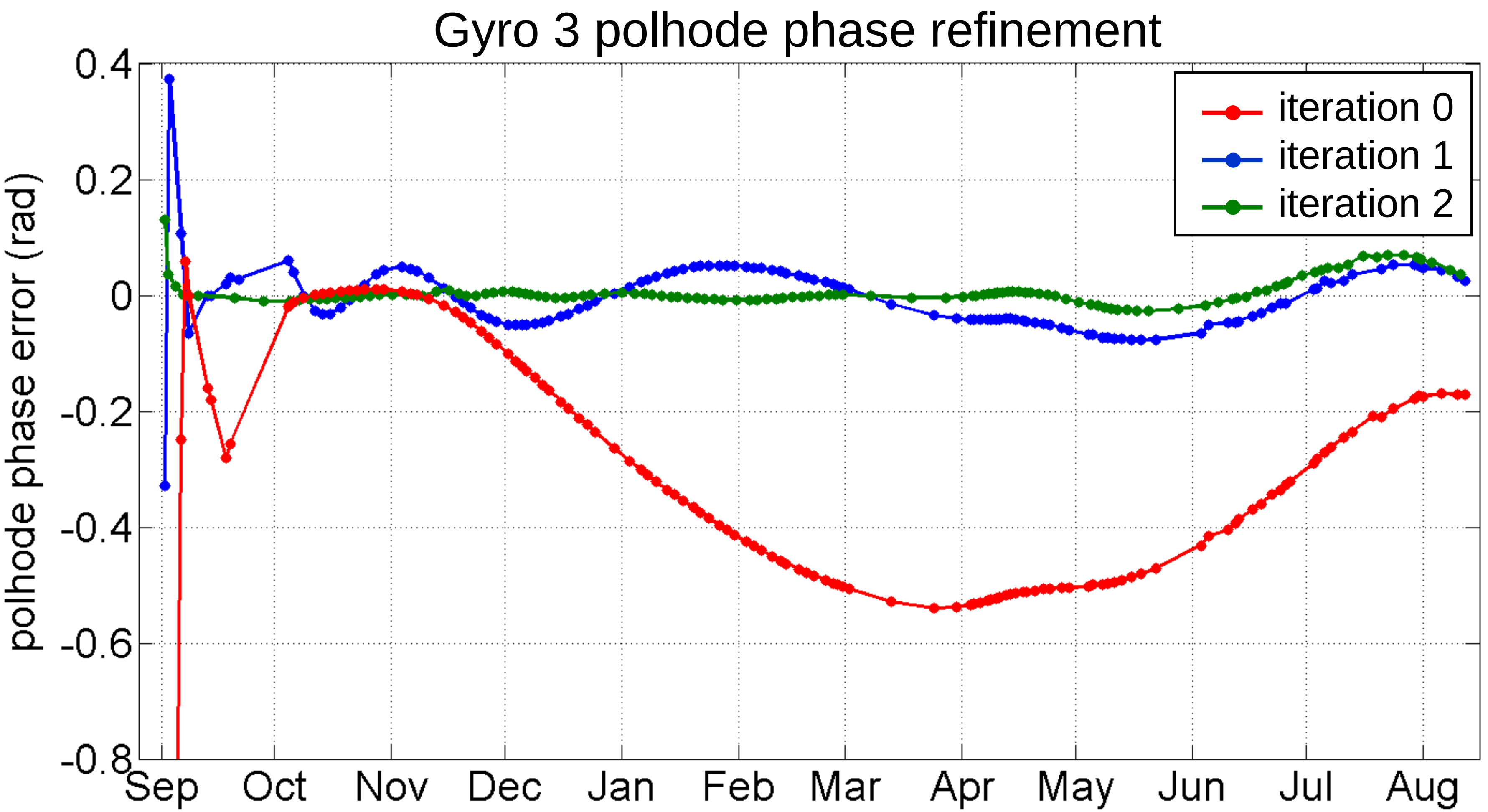


In phase

Provides most accurate estimate of polhode phase

4.10 Iterative Polhode Phase Refinement

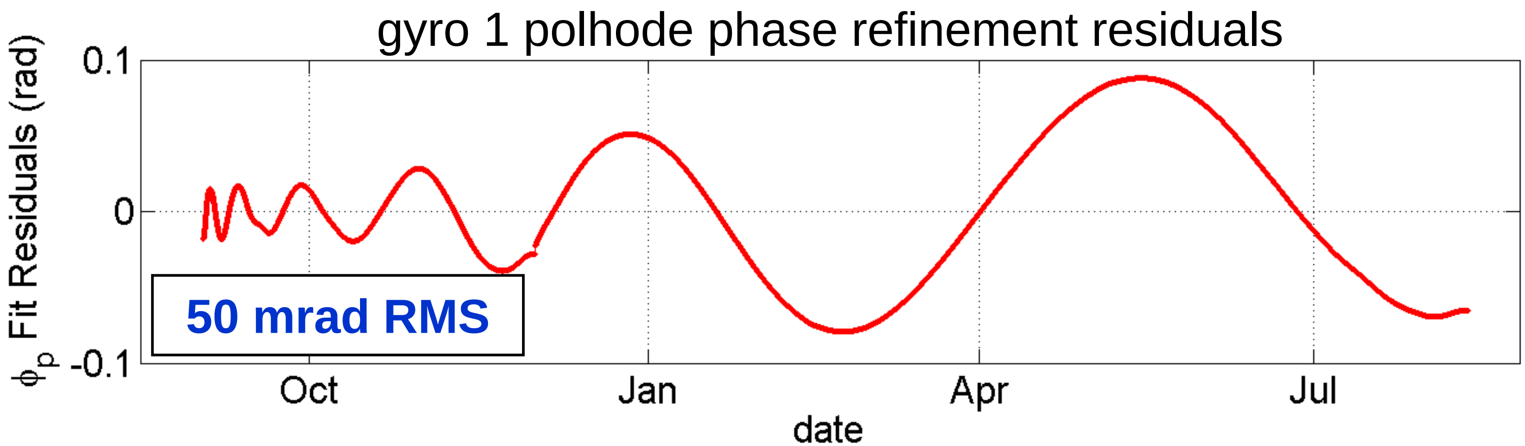
- **With new polhode phase, re-compute spin phase, A_{lm}'s,**



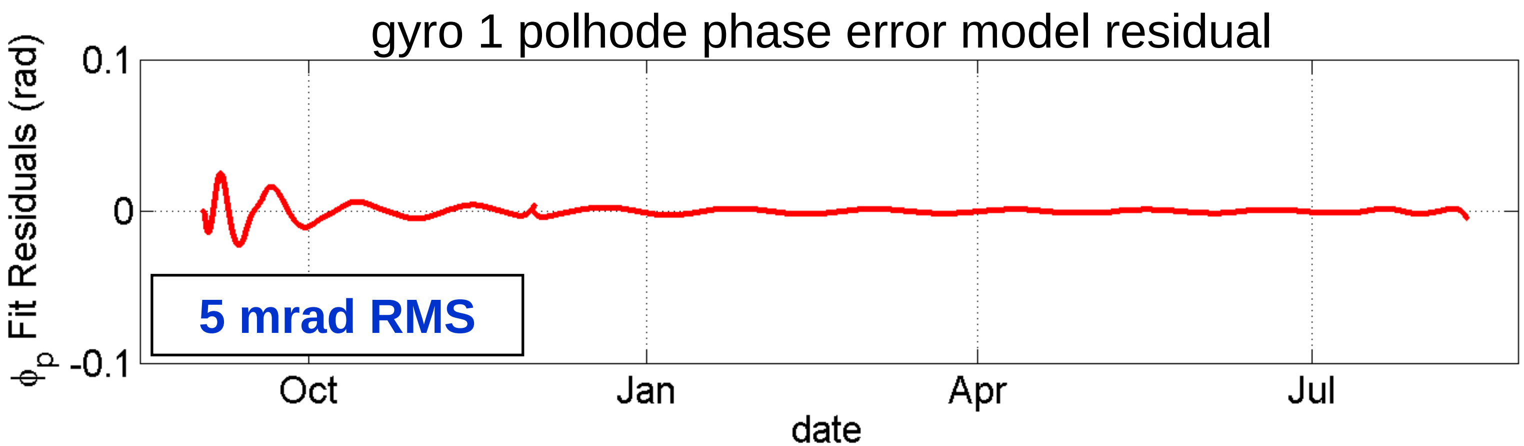
Successive iterations show convergence

4.11 Polhode Phase Error Model (Level C)

- **Polhode phase correction (from $|H_1|$) fit to exp. model**



- **Post-fit residuals fit to Fourier expansion**



1 part in 10^5 fit becomes 1 part in 10^6

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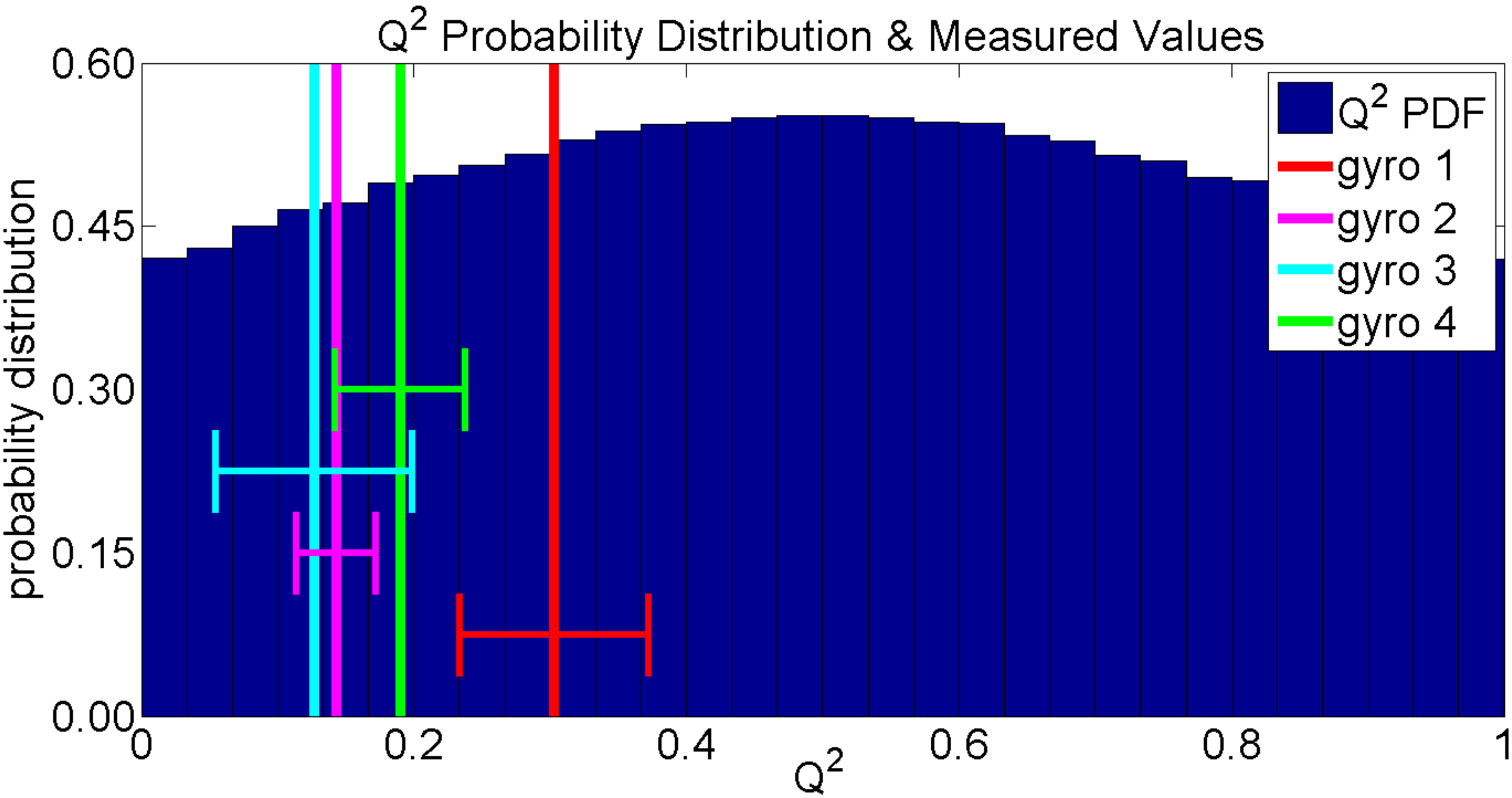
5.1 Rotor Asymmetry Parameter Q^2 (from Level A)

Method	Gyro 1	Gyro 2	Gyro 3	Gyro 4
TFM	0.303 ± 0.069	0.143 ± 0.029	0.127 ± 0.072	0.190 ± 0.048
Previous work	0.33 (0.29 – 0.38)	0.36 (0.14 – 0.43)	~ 0	0.32 (0.30 – 0.40)

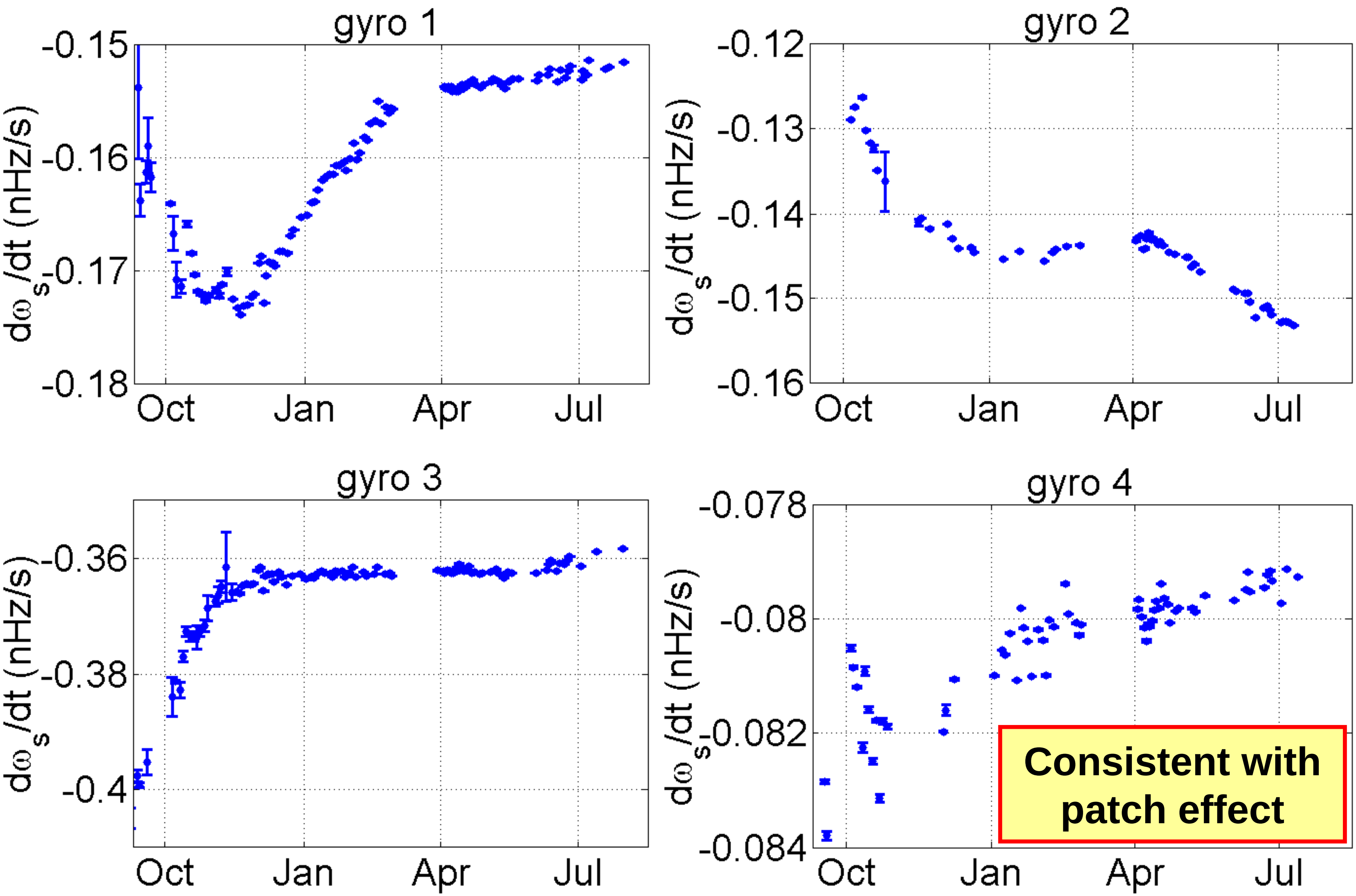
- C_g^{TF} and H_n are relatively insensitive to Q^2
  - Q^2 estimation accurate to ~ 20%
  - Adequate for TFM

5.2 Q^2 Results & Probability Distribution Function

- **Observation:** $0.12 < Q^2 < 0.31$ **all gyros**



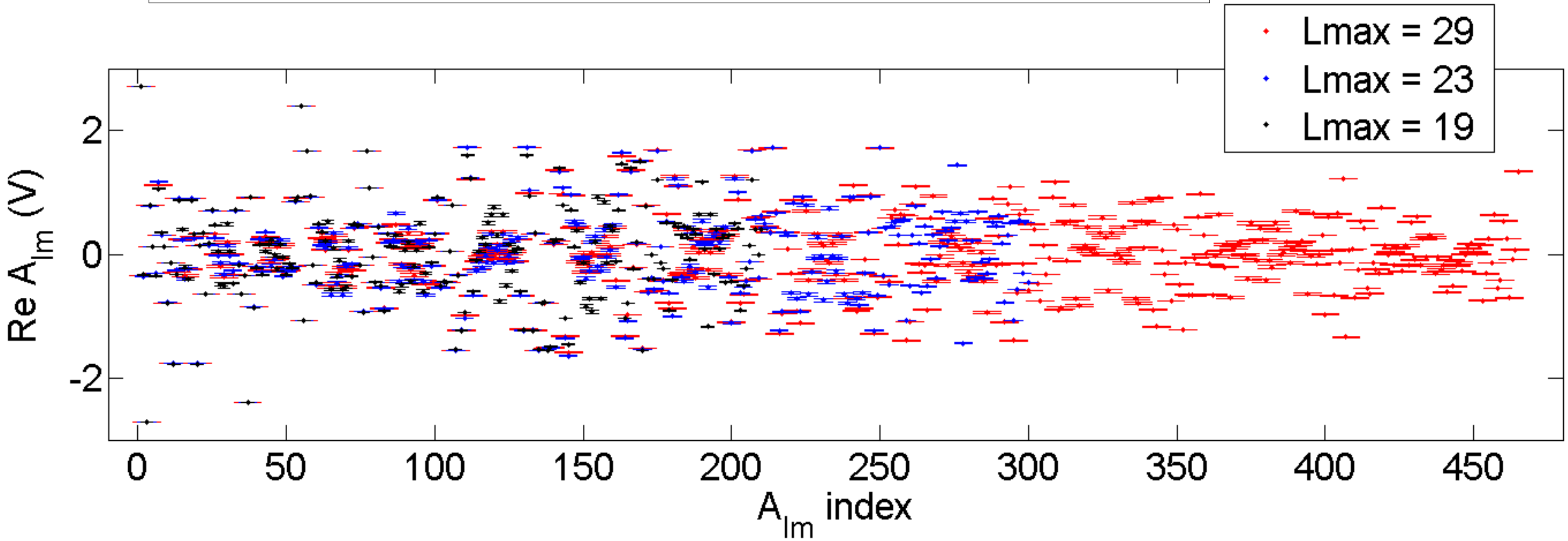
5.3 Spin-Down Rate to ~ 1 pHz/s (from Level B)

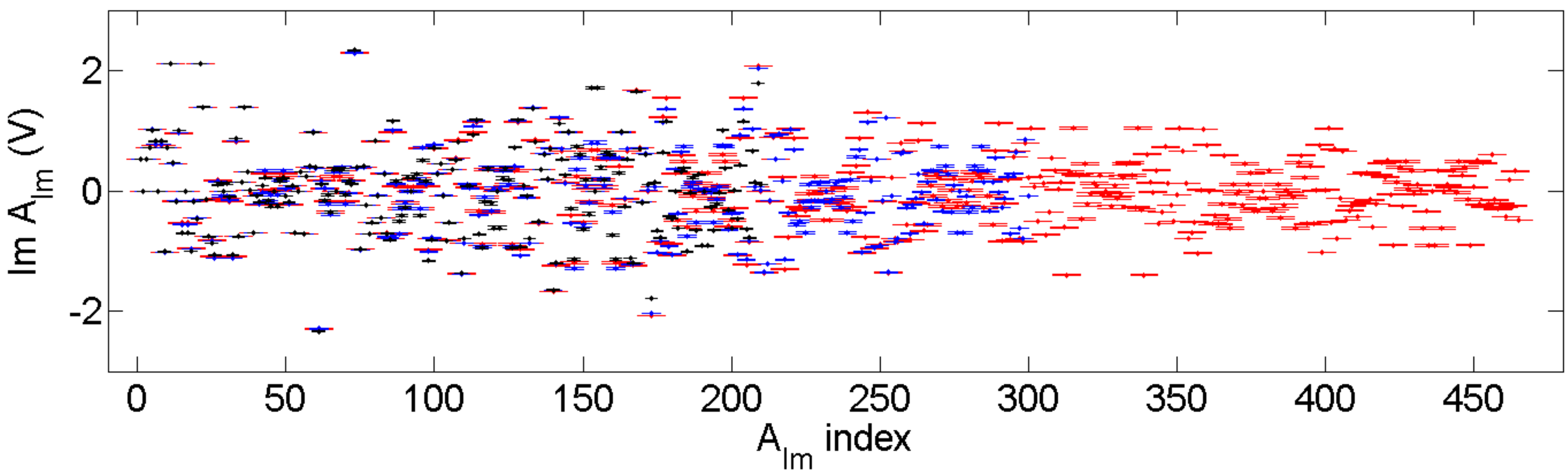


5.4 Spin Speed and Spin-Down Time (from Level B)

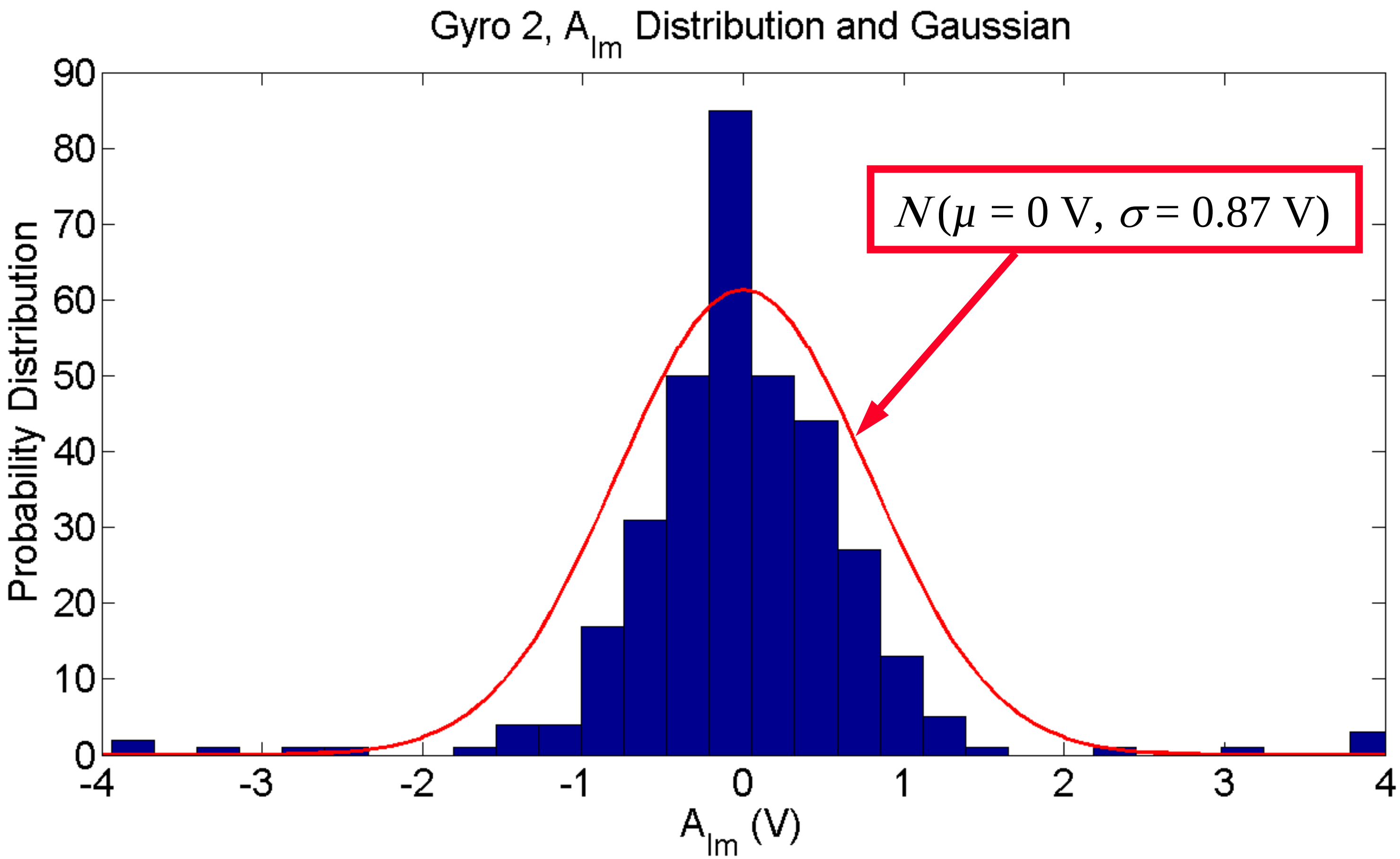
Parameter	Gyro 1	Gyro 2	Gyro 3	Gyro 4
f_s (Hz)	79.40	61.81	82.11	64.84
τ_{sd} (yrs)	15,800	13,400	7,000	25,700

5.5 A_{lm}s for Gyro 1 (from Level C)



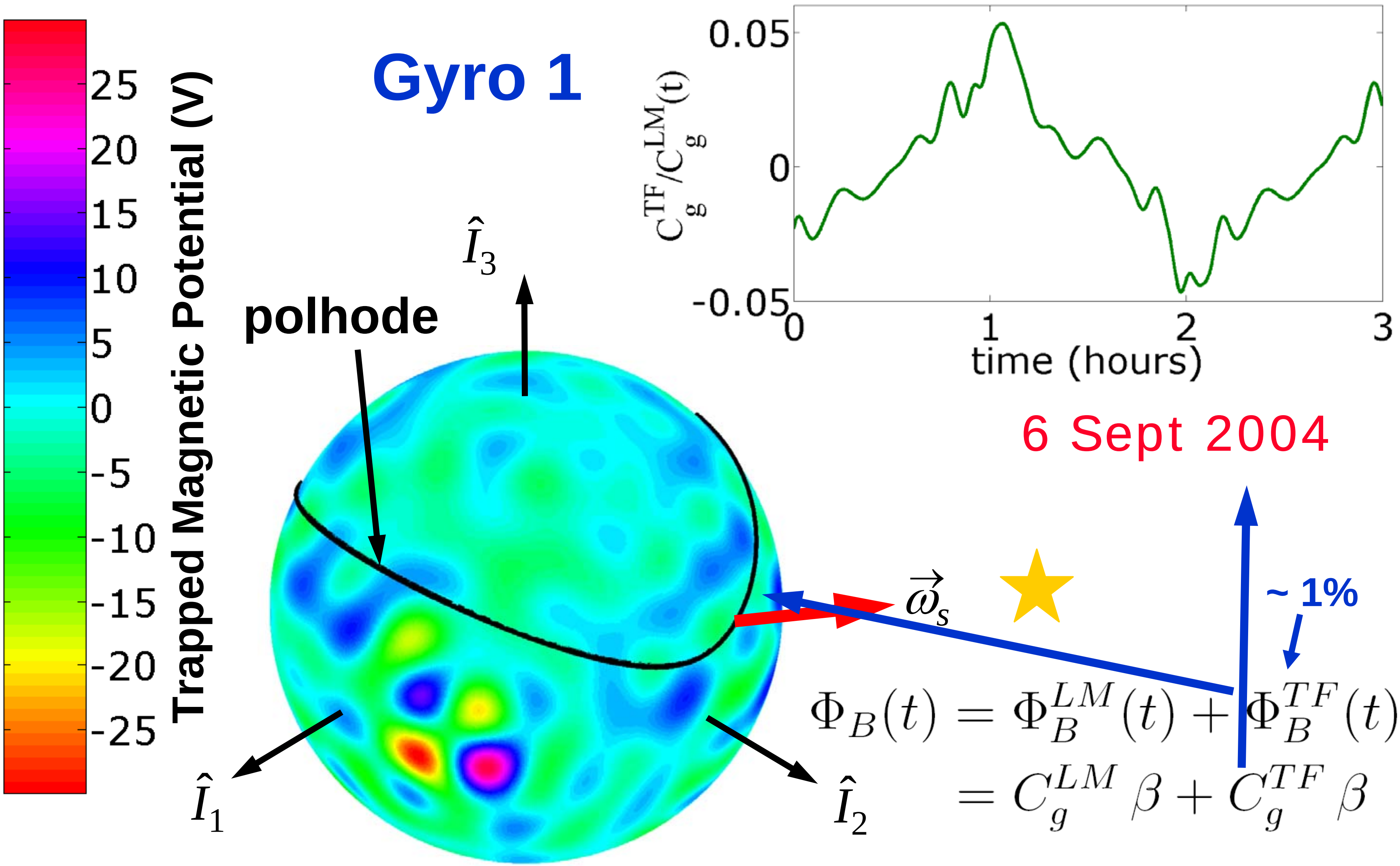


5.6 Distribution of A_{lm} Values



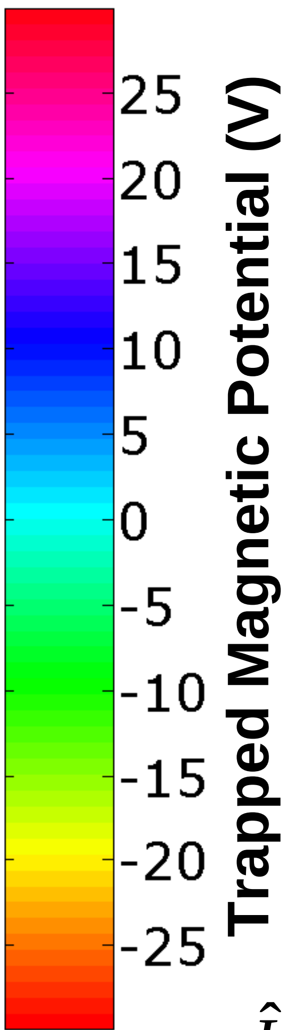
- **Fits indicate A_{lm}s follow zero mean Gaussian distribution, that also agrees with physical understanding of trapped flux**
- **Assuming A_{lm}s normally distributed about zero allowed for more accurate estimates of coefficients with higher indices**

5.7 Trapped Flux & Readout Scale Factor

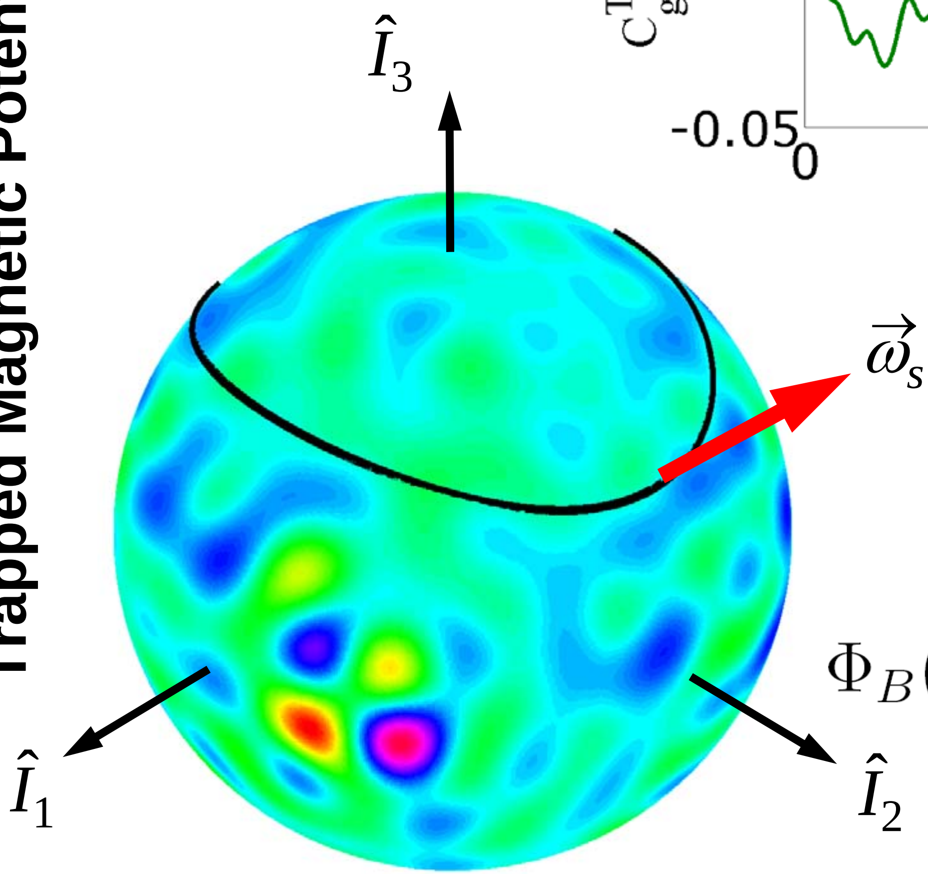


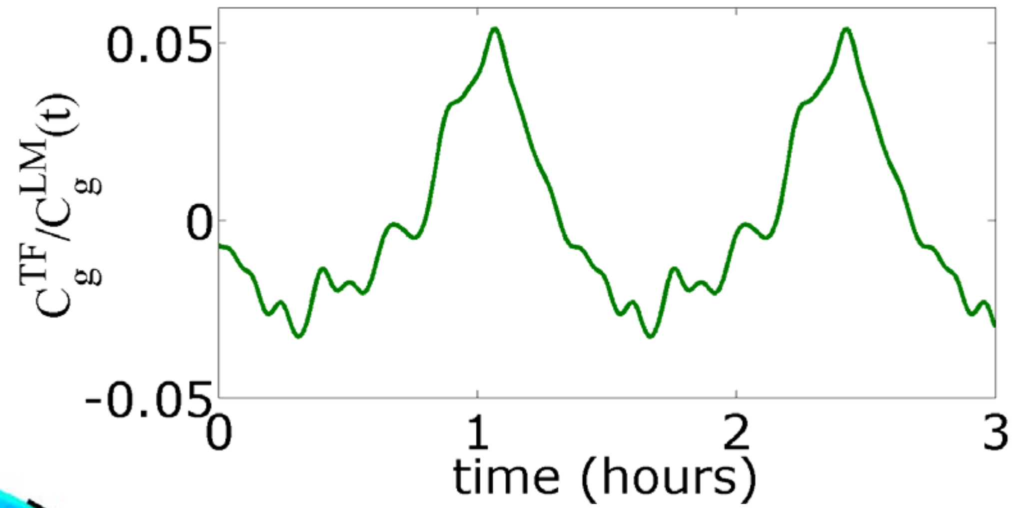
$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.7 Trapped Flux & Readout Scale Factor



Gyro 1

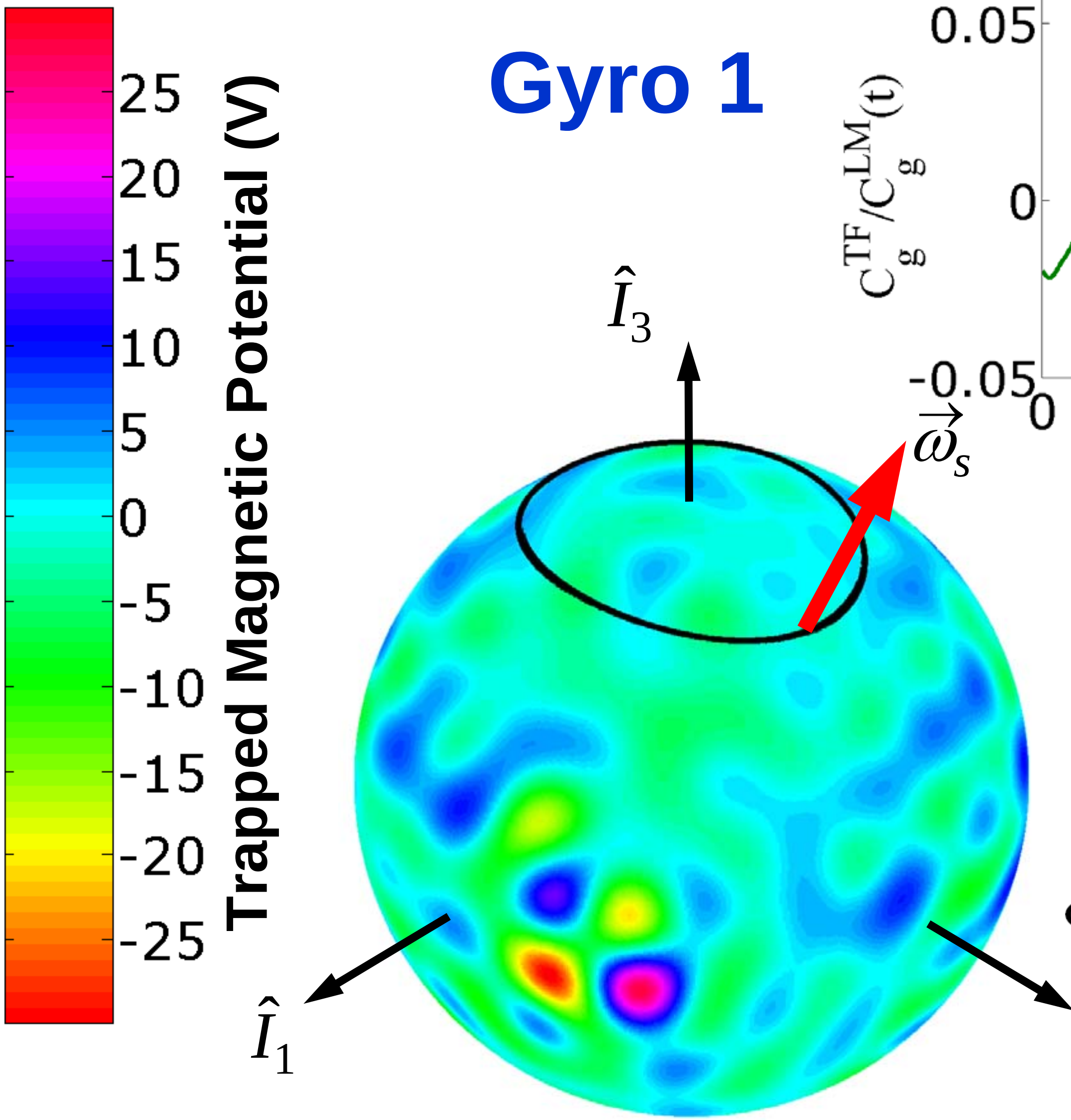


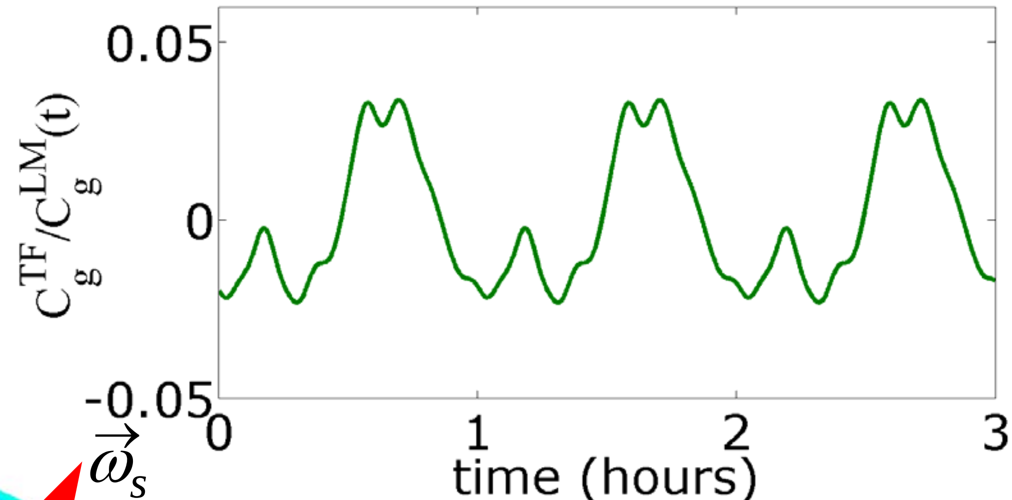


4 Oct 2004

$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.7 Trapped Flux & Readout Scale Factor

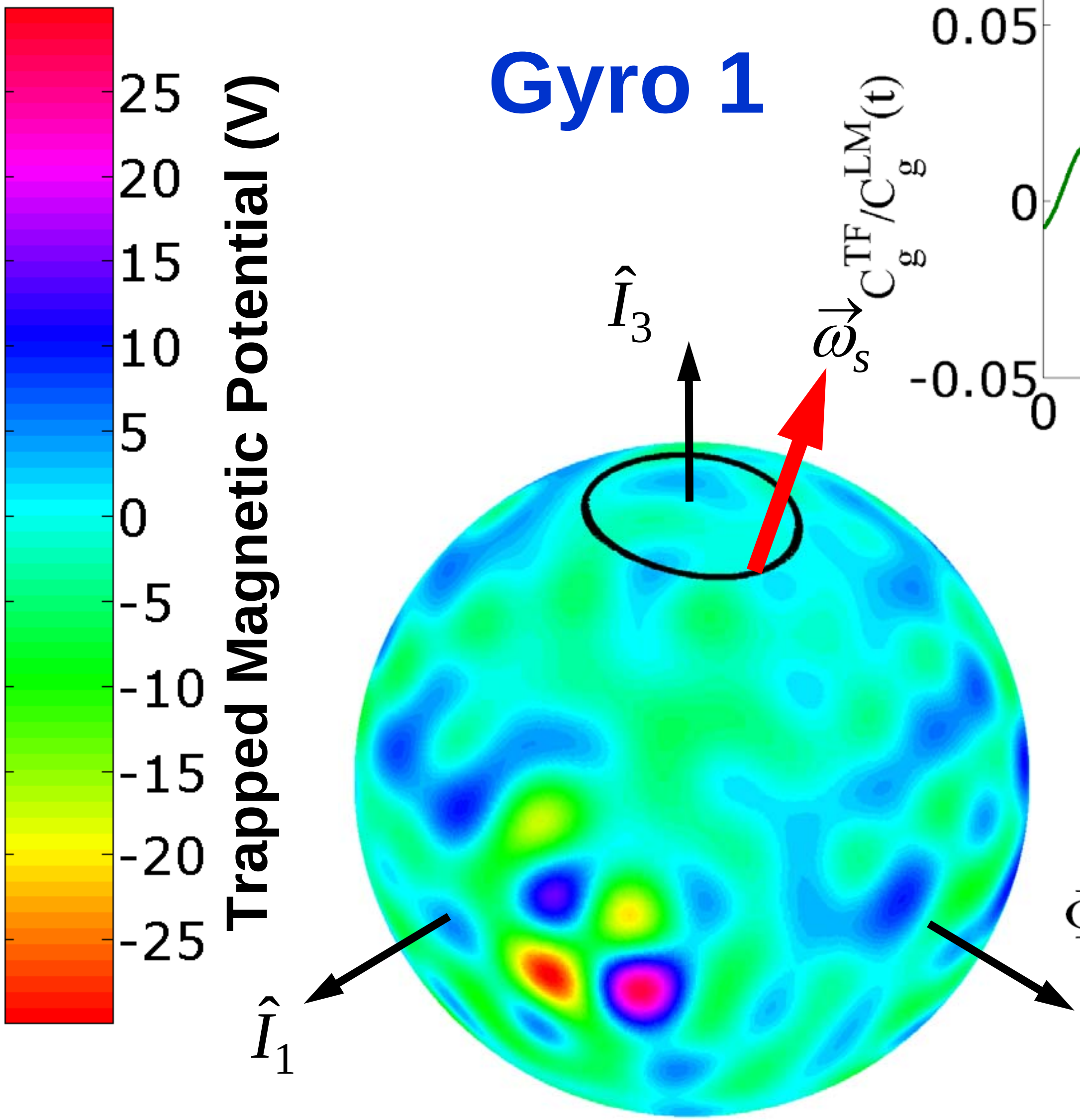


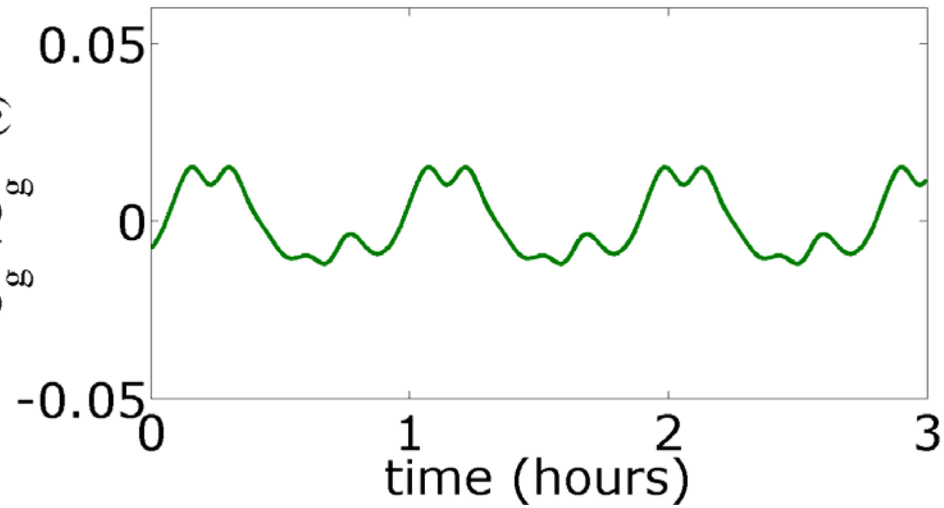


14 Nov 2004

$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.7 Trapped Flux & Readout Scale Factor

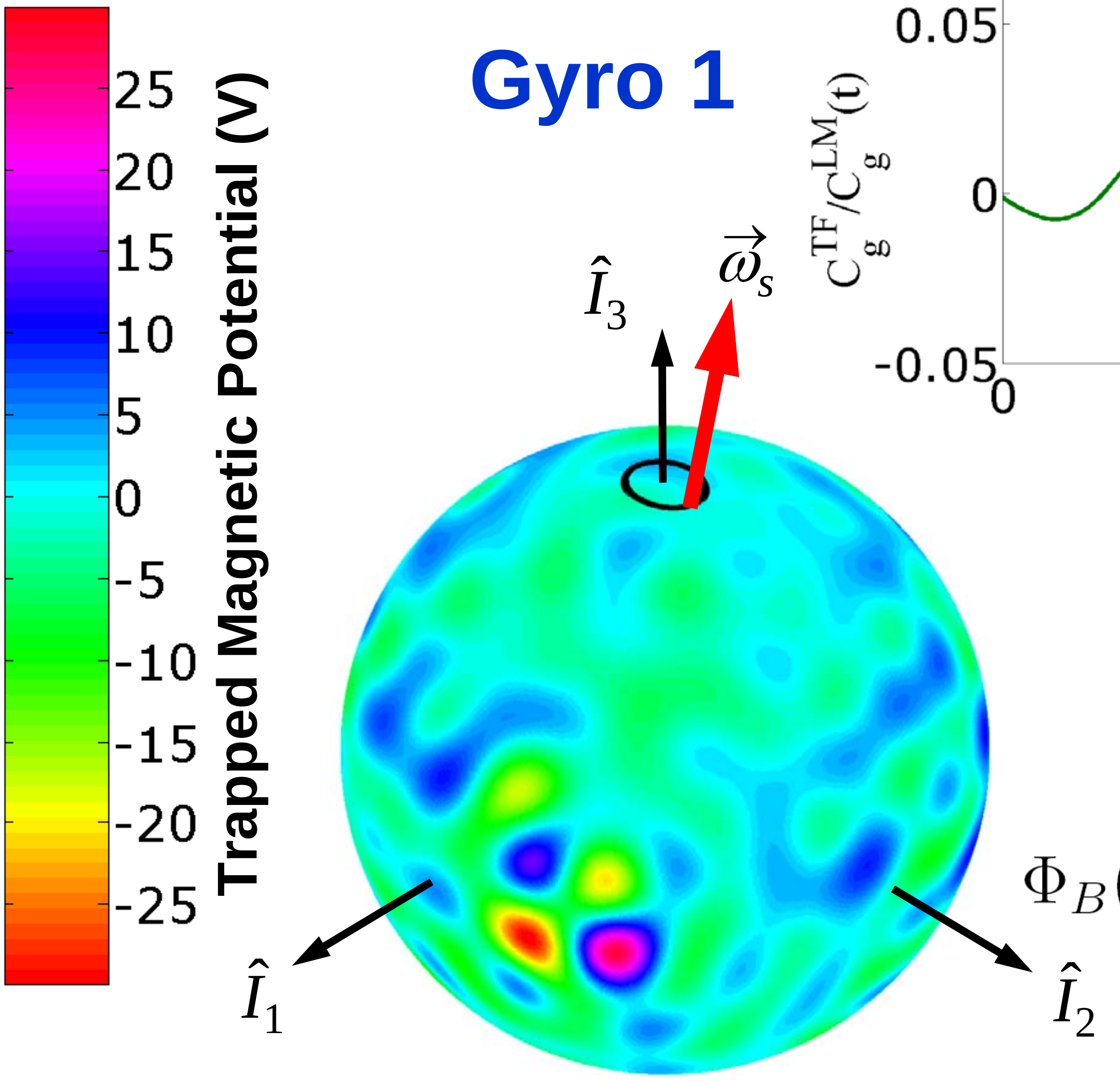


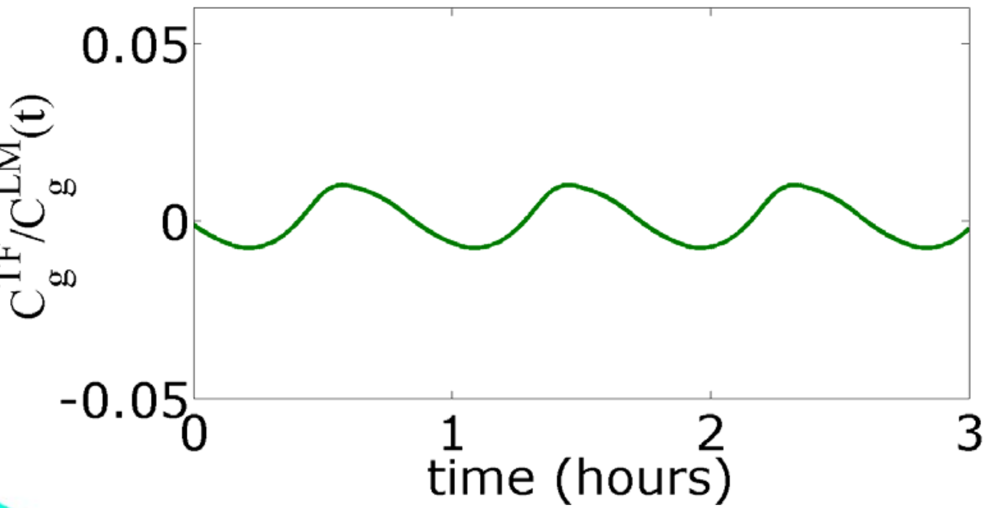


20 Dec 2004

$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.7 Trapped Flux & Readout Scale Factor

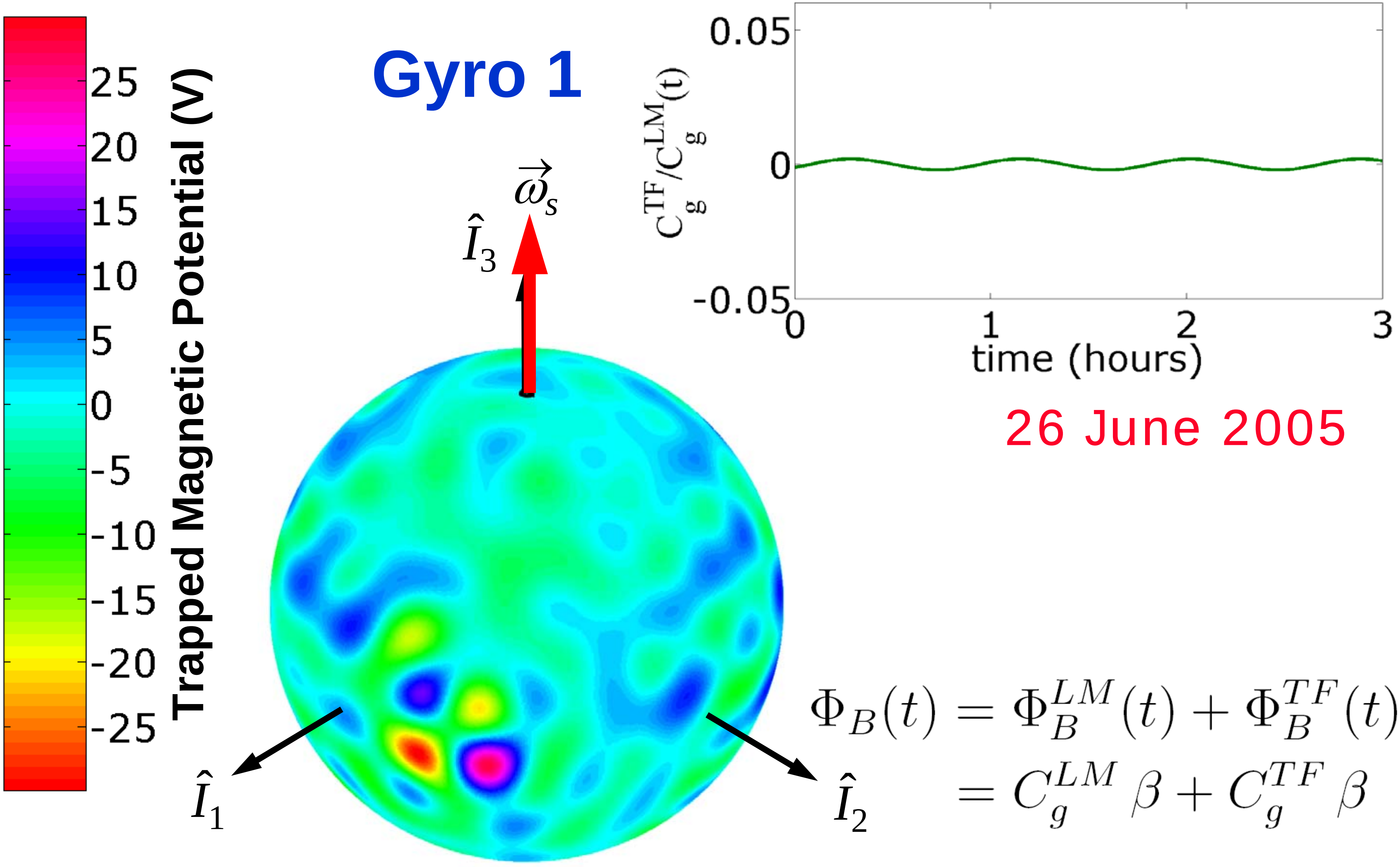




20 Feb 2005

$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.7 Trapped Flux & Readout Scale Factor

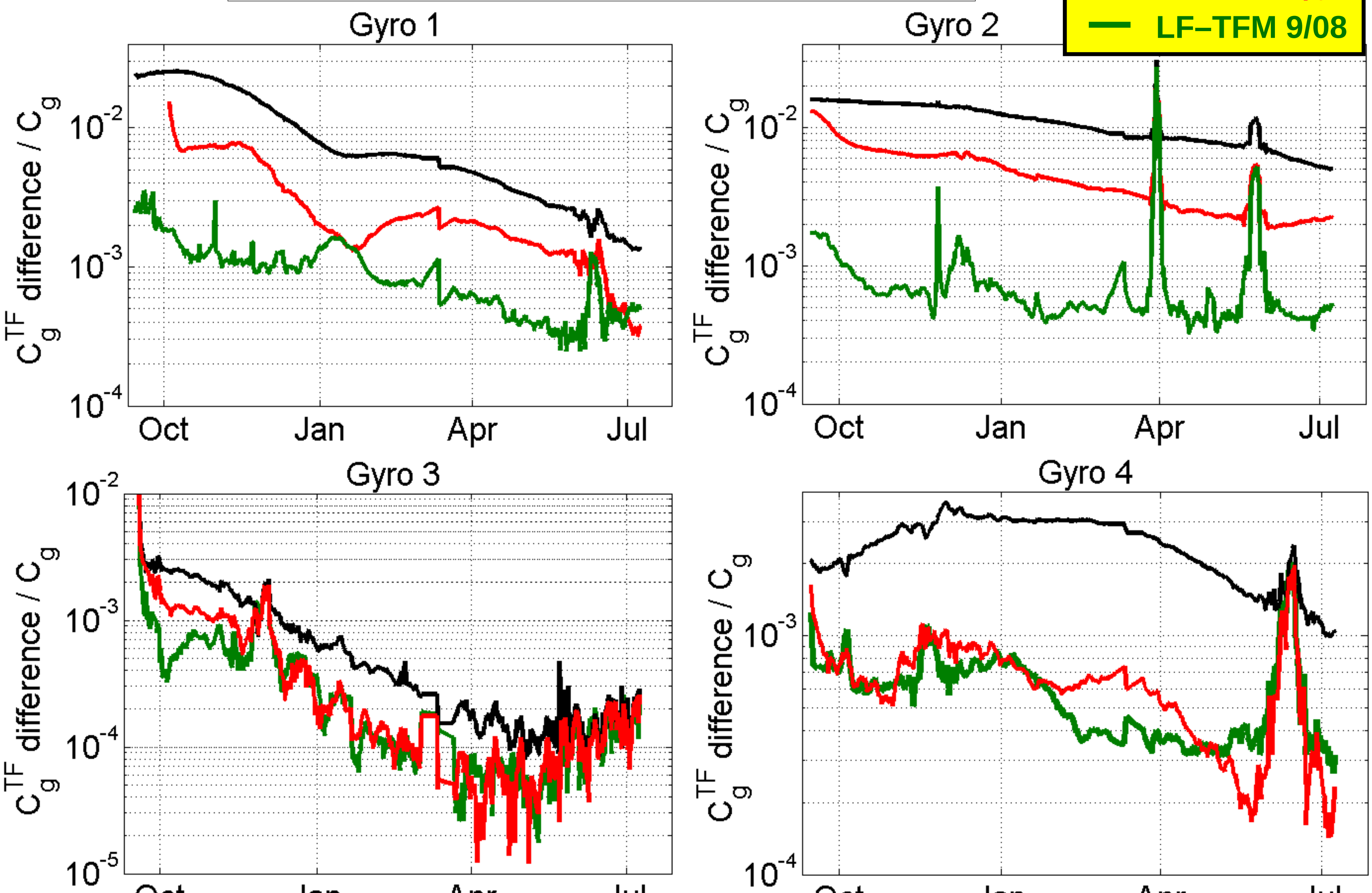


$$\Phi_B(t) = \Phi_B^{LM}(t) + \Phi_B^{TF}(t)$$
$$= C_g^{LM}\,\beta + C_g^{TF}\,\beta$$

5.8 Scale Factor Results, Nov. '07 vs. Aug. '08

Gyro	Data Used	Relative residuals (rms)	Number of Harmonics	Relative Amplitude of Variations	C_g^{TF} Error Relative to C_g (formal sigmas)
1	Oct.	14%	11	3% to 0.2%	0.6×10^{-2} - 2×10^{-2}
	full year	1.1%	21		1.5×10^{-4} - 7.0×10^{-5}
2	Sept. - Dec.	15%	17	1.5% to 0.5%	3×10^{-4} - 6×10^{-4}
	full year	1.5%	25		6.0×10^{-5} - 3.0×10^{-5}
3	Sept. - Dec.	6%	5	1% to 0.01%	3×10^{-3} - 4×10^{-3}
	full year	2.6%	21		2.0×10^{-4} - 1.6×10^{-4}
4	Oct. - Dec.	17%	9	0.3% to 0.1%	3×10^{-3} - 7×10^{-3}
	full year	2.8%	21		8.5×10^{-5} - 6.5×10^{-5}

5.9 TFM & LF C_g Comparison



6. Conclusion. Future Work

- **Polhode period and path change observed on orbit are explained by rotation energy loss and properly analyzed, laying ground for Trapped Flux Mapping**
- **The results of Trapped Flux Mapping based on *odd* harmonics of HF SQUID signal are crucial for getting the best measurement of relativistic drift rate (determining LF scale factor variations and patch effect torque in science analysis)**
- **Future work on examining *even* HF harmonics might lead to new important results, such as:**
  - **Estimation of SQUID signal nonlinearity coefficients**
  - **Alternative science signal, i. e., independent determination of spin–to–pick-up loop misalignment time history**

GP-B Polhode/TFM Task Team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Backup slides …

Elliptic Functions and Parameters in Free Gyro Motion

Exact solution for a free motion of an **asymmetric** rotor is more complicated, since it involves elliptical functions.

Definitions of the relevant **elliptical functions:**

$$K(k) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}}; \qquad \pi/2 = K(0) < K(k) \nearrow K(1) = \infty, \quad 0 < k < 1$$

$$\phi = \int_0^{\mathrm{sn}(\phi,k)} \frac{ds}{\sqrt{(1-s^2)(1-k^2s^2)}} \qquad \text{[transcendental eq. for } \mathrm{sn}(\phi,k), \text{ given } \phi \text{ and } k\text{]}$$

(if $k = 0$, then $\mathrm{sn}(\phi, 0) = \sin\phi$)

$$\mathrm{cn}(\phi,k) = \pm\sqrt{1-\mathrm{sn}^2(\phi,k)}, \qquad \mathrm{dn}(\phi,k) = \sqrt{1-k^2\,\mathrm{sn}^2(\phi,k)}$$

$$\mathrm{tn}(\phi,k) = \mathrm{sn}(\phi,k)/\mathrm{cn}(\phi,k)$$

$$k^2 = \frac{(I_2-I_1)(I_3-I)}{(I_3-I_2)(I-I_1)} = \frac{Q^2}{1-Q^2}\frac{(I_3-I)}{(I-I_1)}, \qquad I = \frac{L^2}{2E}$$

$$T_p = \frac{4K(k)}{\omega_l}\sqrt{\frac{I_1I_2I_3}{I(I_3-I_2)(I-I_1)}}, \quad \omega_l = \frac{2E}{L} = \frac{\vec{\omega}\cdot\vec{L}}{L} = \frac{L}{I}$$

Dissipation Model

- **Euler equation modified for dissipation (unique up to a factor μ):**

$$\frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = -\mu(\omega, E)\,\frac{(\vec{L} \times \vec{\omega}) \times \vec{L}}{L}, \quad \mu > 0$$

- **Dot product with $\vec{L}$ and $\vec{\omega}$ gives, respectively, the angular momentum conservation and the energy evolution equation:**

$$\frac{dE}{dt} = -\mu\left(\omega^2 L^2 - 4E^2\right) \le 0, \quad 2E = \vec{L} \cdot \vec{\omega}$$

- **For GP-B gyros variation of both frequency and energy is very small, so**

$$\mu(\omega, E) = \mu_0 + \mu_{1\,\omega}(\omega - \omega_0) + \mu_{1\,E}(E - E_0) + \ldots \approx \mu_0 = \mathrm{constant}$$

with parameter μ_0 to be estimated from the measured data (e.g., polhode period time history)

Scale Factor Formal Errors